Exercise 4.1

Question 1:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+3+3^2+...+3^{n-1}=\frac{(3^n-1)}{2}$$

Answer

Let the given statement be P(n), i.e.,

P(n): $1 + 3 + 3^2 + ...+ 3^{n-1} = \frac{(3^n-1)}{2}$

For $n = 1$, we have

P(1): $1 = \frac{(3^1-1)}{2}=\frac{3-1}{2}=\frac{2}{2}=1$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1+3+3^2+...+3^{k-1}=\frac{(3^k-1)}{2} \quad ...(i)$$

We shall now prove that P(k + 1) is true.

Consider

$1 + 3 + 3^2 + ... + 3^{k-1} + 3^{(k+1) - 1}$

$= (1 + 3 + 3^2 +... + 3^{k-1}) + 3^k$

$$=\frac{(3^k-1)}{2}+3^k \quad [\text{Using (i)}]$$

$$=\frac{(3^k-1)+2.3^k}{2}$$

$$=\frac{(1+2)3^k-1}{2}$$

$$=\frac{3.3^k-1}{2}$$

$$=\frac{3^{k+1}-1}{2}$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.

Question 2:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$$

Answer

Let the given statement be P(*n*), i.e.,

P(*n*): $1^3 + 2^3 + 3^3 + ... + n^3 = \left(\frac{n(n+1)}{2}\right)^2$

For $n = 1$, we have

P(1): $1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2 = \left(\frac{1.2}{2}\right)^2 = 1^2 = 1$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$1^3 + 2^3 + 3^3 + ... + k^3 = \left(\frac{k(k+1)}{2}\right)^2 \qquad ...(i)$$

We shall now prove that P(*k* + 1) is true.

Consider

$1^3 + 2^3 + 3^3 + ... + k^3 + (k + 1)^3$

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \qquad \left[\text{Using (i)}\right]$$

$$= \frac{k^2(k+1)^2}{4} + (k+1)^3$$

$$= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}$$

$$= \frac{(k+1)^2\left\{k^2 + 4(k+1)\right\}}{4}$$

$$= \frac{(k+1)^2\left\{k^2 + 4k + 4\right\}}{4}$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= \frac{(k+1)^2(k+1+1)^2}{4}$$

$= (1^3 + 2^3 + 3^3 + \dots + k^3) + (k+1)^3$ $= \left(\frac{(k+1)(k+1+1)}{2}\right)^2$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 3:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \dots + \frac{1}{(1+2+3+\dots n)} = \frac{2n}{(n+1)}$$

Answer

Let the given statement be P(n), i.e.,

P(n): $1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots n} = \frac{2n}{n+1}$

For n = 1, we have

P(1): 1 = $\frac{2.1}{1+1} = \frac{2}{2} = 1$ which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1+\frac{1}{1+2}+...+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k}=\frac{2k}{k+1} \qquad ... \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$$1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...+k}+\frac{1}{1+2+3+...+k+(k+1)}$$

$$=\left(1+\frac{1}{1+2}+\frac{1}{1+2+3}+...+\frac{1}{1+2+3+...k}\right)+\frac{1}{1+2+3+...+k+(k+1)}$$

$$=\frac{2k}{k+1}+\frac{1}{1+2+3+...+k+(k+1)} \qquad \text{[Using (i)]}$$

$$=\frac{2k}{k+1}+\frac{1}{\left(\frac{(k+1)(k+1+1)}{2}\right)} \qquad \left[1+2+3+...+n=\frac{n(n+1)}{2}\right]$$

$$=\frac{2k}{(k+1)}+\frac{2}{(k+1)(k+2)}$$

$$=\frac{2}{(k+1)}\left(k+\frac{1}{k+2}\right)$$

Question 4:

Prove the following by using the principle of mathematical induction for all $n \in N$: 1.2.3 + 2.3.4 + … + $n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$

Answer

Let the given statement be P(n), i.e.,

P(n): 1.2.3 + 2.3.4 + … + $n(n + 1)(n + 2) = \frac{n(n+1)(n+2)(n+3)}{4}$

For n = 1, we have

P(1): 1.2.3 = 6 = $\frac{1(1+1)(1+2)(1+3)}{4}=\frac{1.2.3.4}{4}=6$, which is true.

Let P(k) be true for some positive integer k, i.e.,

1.2.3 + 2.3.4 + … + $k(k + 1)(k + 2) = \frac{k(k+1)(k+2)(k+3)}{4} \qquad$ ... (i)

We shall now prove that P(k + 1) is true.

Consider

1.2.3 + 2.3.4 + … + $k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3)$

= {1.2.3 + 2.3.4 + … + $k(k + 1)(k + 2)$} + $(k + 1)(k + 2)(k + 3)$

$$=\frac{k(k+1)(k+2)(k+3)}{4}+(k+1)(k+2)(k+3) \qquad [\text{Using (i)}]$$

$$=(k+1)(k+2)(k+3)\left(\frac{k}{4}+1\right)$$

$$=\frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$=\frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 5:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.3+2.3^2+3.3^3+\ldots+n.3^n=\frac{(2n-1)3^{n+1}+3}{4}$$

Answer

Let the given statement be P(n), i.e.,

P(n) : $1.3+2.3^2+3.3^3+\ldots+n3^n=\frac{(2n-1)3^{n+1}+3}{4}$

For n = 1, we have

P(1): 1.3 = 3 $=\frac{(2.1-1)3^{1+1}+3}{4}=\frac{3^2+3}{4}=\frac{12}{4}=3$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$1.3+2.3^2+3.3^3+\ldots+k3^k=\frac{(2k-1)3^{k+1}+3}{4} \qquad \ldots \text{(i)}$$

We shall now prove that P(k + 1) is true.

Consider

$1.3 + 2.3^2 + 3.3^3 + \ldots + k3^k + (k + 1)\ 3^{k+1}$

$= (1.3 + 2.3^2 + 3.3^3 + \ldots + k.3^k) + (k + 1)\ 3^{k+1}$

$$= \frac{(2k-1)3^{k+1}+3}{4} + (k+1)3^{k+1} \qquad [\text{Using (i)}]$$

$$= \frac{(2k-1)3^{k+1}+3+4(k+1)3^{k+1}}{4}$$

$$= \frac{3^{k+1}\{2k-1+4(k+1)\}+3}{4}$$

$$= \frac{3^{k+1}\{6k+3\}+3}{4}$$

$$= \frac{3^{k+1}.3\{2k+1\}+3}{4}$$

$$= \frac{3^{(k+1)+1}\{2k+1\}+3}{4}$$

$$= \frac{\{2(k+1)-1\}3^{(k+1)+1}+3}{4}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 6:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$$

Answer

Let the given statement be P(n), i.e.,

P(n): $1.2+2.3+3.4+\ldots+n.(n+1)=\left[\frac{n(n+1)(n+2)}{3}\right]$

For n = 1, we have

P(1): $1.2=2=\frac{1(1+1)(1+2)}{3}=\frac{1.2.3}{3}=2$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$1.2+2.3+3.4+.....+k.(k+1)=\left[\frac{k(k+1)(k+2)}{3}\right] \quad ...\text{(i)}$$

We shall now prove that P(*k* + 1) is true.

Consider

1.2 + 2.3 + 3.4 + … + *k*.(*k* + 1) + (*k* + 1).(*k* + 2)

= [1.2 + 2.3 + 3.4 + … + *k*.(*k* + 1)] + (*k* + 1).(*k* + 2)

$$=\frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \quad \text{[Using (i)]}$$

$$=(k+1)(k+2)\left(\frac{k}{3}+1\right)$$

$$=\frac{(k+1)(k+2)(k+3)}{3}$$

$$=\frac{(k+1)(k+1+1)(k+1+2)}{3}$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.

Question 7:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$$

Answer

Let the given statement be P(*n*), i.e.,

P(*n*): $1.3+3.5+5.7+...+(2n-1)(2n+1)=\frac{n(4n^2+6n-1)}{3}$

For *n* = 1, we have

$\text{P}(1):1.3=3=\frac{1(4.1^2+6.1-1)}{3}=\frac{4+6-1}{3}=\frac{9}{3}=3$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$1.3+3.5+5.7+......+(2k-1)(2k+1)=\frac{k(4k^2+6k-1)}{3} \qquad ...(i)$$

We shall now prove that P(k + 1) is true.

Consider

(1.3 + 3.5 + 5.7 + … + (2k − 1) (2k + 1) + {2(k + 1) − 1}{2(k + 1) + 1}

$$=\frac{k(4k^2+6k-1)}{3}+(2k+2-1)(2k+2+1) \qquad [\text{Using (i)}]$$

$$=\frac{k(4k^2+6k-1)}{3}+(2k+1)(2k+3)$$

$$=\frac{k(4k^2+6k-1)}{3}+(4k^2+8k+3)$$

$$=\frac{k(4k^2+6k-1)+3(4k^2+8k+3)}{3}$$

$$=\frac{4k^3+6k^2-k+12k^2+24k+9}{3}$$

$$=\frac{4k^3+18k^2+23k+9}{3}$$

$$=\frac{4k^3+14k^2+9k+4k^2+14k+9}{3}$$

$$=\frac{k(4k^2+14k+9)+1(4k^2+14k+9)}{3}$$

$$=\frac{(k+1)(4k^2+14k+9)}{3}$$

$$=\frac{(k+1)\{4k^2+8k+4+6k+6-1\}}{3}$$

$$=\frac{(k+1)\{4(k^2+2k+1)+6(k+1)-1\}}{3}$$

$$=\frac{(k+1)\{4(k+1)^2+6(k+1)-1\}}{3}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 8:

Prove the following by using the principle of mathematical induction for all $n \in N$: 1.2 + $2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\ 2^{n+1} + 2$

Answer

Let the given statement be P(n), i.e.,

P(n): $1.2 + 2.2^2 + 3.2^2 + \ldots + n.2^n = (n - 1)\ 2^{n+1} + 2$

For $n = 1$, we have

P(1): $1.2 = 2 = (1 - 1)\ 2^{1+1} + 2 = 0 + 2 = 2$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$1.2 + 2.2^2 + 3.2^2 + \ldots + k.2^k = (k - 1)\ 2^{k+1} + 2$ … (i)

We shall now prove that P(k + 1) is true.

Consider

$$\left\{1.2 + 2.2^2 + 3.2^3 + \ldots + k.2^k\right\} + (k+1)\cdot 2^{k+1}$$
$$= (k-1)2^{k+1} + 2 + (k+1)2^{k+1}$$
$$= 2^{k+1}\left\{(k-1) + (k+1)\right\} + 2$$
$$= 2^{k+1}.2k + 2$$
$$= k.2^{(k+1)+1} + 2$$
$$= \left\{(k+1) - 1\right\}2^{(k+1)+1} + 2$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 9:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Answer

Let the given statement be P(n), i.e.,

P(n): $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$

For n = 1, we have

P(1): $\frac{1}{2}=1-\frac{1}{2^1}=\frac{1}{2}$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+....+\frac{1}{2^k}=1-\frac{1}{2^k} \quad ...(i)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+.......+\frac{1}{2^k}\right)+\frac{1}{2^{k+1}}$$

$$=\left(1-\frac{1}{2^k}\right)+\frac{1}{2^{k+1}} \quad [\text{Using (i)}]$$

$$=1-\frac{1}{2^k}+\frac{1}{2.2^k}$$

$$=1-\frac{1}{2^k}\left(1-\frac{1}{2}\right)$$

$$=1-\frac{1}{2^k}\left(\frac{1}{2}\right)$$

$$=1-\frac{1}{2^{k+1}}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 10:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$$

Answer

Let the given statement be P(n), i.e.,

P(n): $\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3n-1)(3n+2)}=\frac{n}{(6n+4)}$

For n = 1, we have

$$P(1)=\frac{1}{2.5}=\frac{1}{10}=\frac{1}{6.1+4}=\frac{1}{10}$$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+...+\frac{1}{(3k-1)(3k+2)}=\frac{k}{6k+4} \quad ... (i)$$

We shall now prove that P(*k* + 1) is true.

Consider

$$\frac{1}{2.5}+\frac{1}{5.8}+\frac{1}{8.11}+.......+\frac{1}{(3k-1)(3k+2)}+\frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}}$$

$$=\frac{k}{6k+4}+\frac{1}{(3k+3-1)(3k+3+2)} \quad [\text{Using (i)}]$$

$$=\frac{k}{6k+4}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{k}{2(3k+2)}+\frac{1}{(3k+2)(3k+5)}$$

$$=\frac{1}{(3k+2)}\left(\frac{k}{2}+\frac{1}{3k+5}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{k(3k+5)+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{3k^2+5k+2}{2(3k+5)}\right)$$

$$=\frac{1}{(3k+2)}\left(\frac{(3k+2)(k+1)}{2(3k+5)}\right)$$

$$=\frac{(k+1)}{6k+10}$$

$$=\frac{(k+1)}{6(k+1)+4}$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.

Question 11:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+...+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

Answer

Let the given statement be P(n), i.e.,

P(n): $$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+...+\frac{1}{n(n+1)(n+2)}=\frac{n(n+3)}{4(n+1)(n+2)}$$

For n = 1, we have

$$\text{P}(1):\frac{1}{1\cdot2\cdot3}=\frac{1\cdot(1+3)}{4(1+1)(1+2)}=\frac{1\cdot4}{4\cdot2\cdot3}=\frac{1}{1\cdot2\cdot3}$$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$$\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+...+\frac{1}{k(k+1)(k+2)}=\frac{k(k+3)}{4(k+1)(k+2)} \qquad ...(\text{i})$$

We shall now prove that P(k + 1) is true.

Consider

$$\left[\frac{1}{1\cdot2\cdot3}+\frac{1}{2\cdot3\cdot4}+\frac{1}{3\cdot4\cdot5}+.....+\frac{1}{k(k+1)(k+2)}\right]+\frac{1}{(k+1)(k+2)(k+3)}$$

$$=\frac{k(k+3)}{4(k+1)(k+2)}+\frac{1}{(k+1)(k+2)(k+3)} \qquad [\text{Using (i)}]$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)}{4}+\frac{1}{k+3}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+3)^2+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+6k+9)+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+6k^2+9k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k^3+2k^2+k+4k^2+8k+4}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k^2+2k+1)+4(k^2+2k+1)}{4(k+3)}\right\}$$

$$=\frac{1}{(k+1)(k+2)}\left\{\frac{k(k+1)^2+4(k+1)^2}{4(k+3)}\right\}$$

$$=\frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)}$$

$$=\frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 12:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

Answer

Let the given statement be P(*n*), i.e.,

$$P(n): a + ar + ar^2 + ... + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

For $n = 1$, we have

$$P(1): a = \frac{a(r^1 - 1)}{(r - 1)} = a$$

, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$a + ar + ar^2 + ... + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \quad ... (i)$$

We shall now prove that P(*k* + 1) is true.

Consider

$$\{a + ar + ar^2 + ... + ar^{k-1}\} + ar^{(k+1)-1}$$

$$= \frac{a(r^k - 1)}{r - 1} + ar^k \quad [\text{Using (i)}]$$

$$= \frac{a(r^k - 1) + ar^k (r - 1)}{r - 1}$$

$$= \frac{a(r^k - 1) + ar^{k+1} - ar^k}{r - 1}$$

$$= \frac{ar^k - a + ar^{k+1} - ar^k}{r - 1}$$

$$= \frac{ar^{k+1} - a}{r - 1}$$

$$= \frac{a(r^{k+1} - 1)}{r - 1}$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.

Question 13:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

Answer

Let the given statement be P(*n*), i.e.,

$$P(n):\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2n+1)}{n^2}\right)=(n+1)^2$$

For *n* = 1, we have

$$P(1):\left(1+\frac{3}{1}\right)=4=(1+1)^2=2^2=4, \text{ which is true.}$$

Let P(*k*) be true for some positive integer *k*, i.e.,

$$\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)=(k+1)^2 \quad \ldots (1)$$

We shall now prove that P(*k* + 1) is true.

Consider

$$\left[\left(1+\frac{3}{1}\right)\left(1+\frac{5}{4}\right)\left(1+\frac{7}{9}\right)\ldots\left(1+\frac{(2k+1)}{k^2}\right)\right]\left\{1+\frac{\{2(k+1)+1\}}{(k+1)^2}\right\}$$

$$=(k+1)^2\left(1+\frac{2(k+1)+1}{(k+1)^2}\right) \quad [\text{Using}(1)]$$

$$=(k+1)^2\left[\frac{(k+1)^2+2(k+1)+1}{(k+1)^2}\right]$$

$$=(k+1)^2+2(k+1)+1$$

$$=\{(k+1)+1\}^2$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.

Question 14:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right)=(n+1)$$

Answer

Let the given statement be P(*n*), i.e.,

$$P(n):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{n}\right)=(n+1)$$

For *n* = 1, we have

$$P(1):\left(1+\frac{1}{1}\right)=2=(1+1)$$

, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$P(k):\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{k}\right)=(k+1) \qquad ...(1)$$

We shall now prove that P(*k* + 1) is true.

Consider

$$\left[\left(1+\frac{1}{1}\right)\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)...\left(1+\frac{1}{k}\right)\right]\left(1+\frac{1}{k+1}\right)$$

$$=(k+1)\left(1+\frac{1}{k+1}\right) \qquad [\text{Using } (1)]$$

$$=(k+1)\left(\frac{(k+1)+1}{(k+1)}\right)$$

$$=(k+1)+1$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.

Question 15:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

Answer

Let the given statement be P(n), i.e.,

$$P(n)=1^2+3^2+5^2+...+(2n-1)^2=\frac{n(2n-1)(2n+1)}{3}$$

For $n=1$, we have

$$P(1)=1^2=1=\frac{1(2.1-1)(2.1+1)}{3}=\frac{1.1.3}{3}=1,\text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$P(k)=1^2+3^2+5^2+...+(2k-1)^2=\frac{k(2k-1)(2k+1)}{3} \quad ...(1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left\{1^2+3^2+5^2+...+(2k-1)^2\right\}+\left\{2(k+1)-1\right\}^2$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+2-1)^2 \quad [\text{Using } (1)]$$

$$=\frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

$$=\frac{k(2k-1)(2k+1)+3(2k+1)^2}{3}$$

$$=\frac{(2k+1)\left\{k(2k-1)+3(2k+1)\right\}}{3}$$

$$=\frac{(2k+1)\left\{2k^2-k+6k+3\right\}}{3}$$

$$=\frac{(2k+1)\{2k^2+5k+3\}}{3}$$

$$=\frac{(2k+1)\{2k^2+2k+3k+3\}}{3}$$

$$=\frac{(2k+1)\{2k(k+1)+3(k+1)\}}{3}$$

$$=\frac{(2k+1)(k+1)(2k+3)}{3}$$

$$=\frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 16:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

Answer

Let the given statement be P(n), i.e.,

$$P(n):\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3n-2)(3n+1)}=\frac{n}{(3n+1)}$$

For $n=1$, we have

$$P(1)=\frac{1}{1.4}=\frac{1}{3.1+1}=\frac{1}{4}=\frac{1}{1.4}, \text{ which is true.}$$

Let P(k) be true for some positive integer k, i.e.,

$$P(k)=\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+\ldots+\frac{1}{(3k-2)(3k+1)}=\frac{k}{3k+1} \quad \ldots (1)$$

We shall now prove that P(k + 1) is true.

Consider

$$\left\{\frac{1}{1.4}+\frac{1}{4.7}+\frac{1}{7.10}+...+\frac{1}{(3k-2)(3k+1)}\right\}+\frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}}$$

$$=\frac{k}{3k+1}+\frac{1}{(3k+1)(3k+4)} \qquad [\text{Using } (1)]$$

$$=\frac{1}{(3k+1)}\left\{k+\frac{1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{k(3k+4)+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+4k+1}{(3k+4)}\right\}$$

$$=\frac{1}{(3k+1)}\left\{\frac{3k^2+3k+k+1}{(3k+4)}\right\}$$

$$=\frac{(3k+1)(k+1)}{(3k+1)(3k+4)}$$

$$=\frac{(k+1)}{3(k+1)+1}$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.

Question 17:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

Answer

Let the given statement be P(*n*), i.e.,

$$\text{P}(n):\frac{1}{3.5}+\frac{1}{5.7}+\frac{1}{7.9}+...+\frac{1}{(2n+1)(2n+3)}=\frac{n}{3(2n+3)}$$

For *n* = 1, we have

$$P(1): \frac{1}{3.5} = \frac{1}{3(2.1+3)} = \frac{1}{3.5}$$, which is true.

Let P(*k*) be true for some positive integer *k*, i.e.,

$$P(k): \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + ... + \frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)} \quad ... (1)$$

We shall now prove that P(*k* + 1) is true.

Consider

$$\left[\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + ... + \frac{1}{(2k+1)(2k+3)}\right] + \frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}}$$

$$= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)} \quad [\text{Using } (1)]$$

$$= \frac{1}{(2k+3)}\left[\frac{k}{3} + \frac{1}{(2k+5)}\right]$$

$$= \frac{1}{(2k+3)}\left[\frac{k(2k+5)+3}{3(2k+5)}\right]$$

$$= \frac{1}{(2k+3)}\left[\frac{2k^2+5k+3}{3(2k+5)}\right]$$

$$= \frac{1}{(2k+3)}\left[\frac{2k^2+2k+3k+3}{3(2k+5)}\right]$$

$$= \frac{1}{(2k+3)}\left[\frac{2k(k+1)+3(k+1)}{3(2k+5)}\right]$$

$$= \frac{(k+1)(2k+3)}{3(2k+3)(2k+5)}$$

$$= \frac{(k+1)}{3\{2(k+1)+3\}}$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.

Question 18:

Prove the following by using the principle of mathematical induction for all $n \in N$:

$$1+2+3+...+n<\frac{1}{8}(2n+1)^2$$

Answer

Let the given statement be P(n), i.e.,

$$\text{P}(n):1+2+3+...+n<\frac{1}{8}(2n+1)^2$$

It can be noted that P(n) is true for n = 1 since $1<\frac{1}{8}(2.1+1)^2=\frac{9}{8}$.

Let P(k) be true for some positive integer k, i.e.,

$$1+2+...+k<\frac{1}{8}(2k+1)^2 \qquad ... (1)$$

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$(1+2+...+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1) \qquad [\text{Using}(1)]$$

$$<\frac{1}{8}\left\{(2k+1)^2+8(k+1)\right\}$$

$$<\frac{1}{8}\left\{4k^2+4k+1+8k+8\right\}$$

$$<\frac{1}{8}\left\{4k^2+12k+9\right\}$$

$$<\frac{1}{8}(2k+3)^2$$

$$<\frac{1}{8}\left\{2(k+1)+1\right\}^2$$

Hence, $(1+2+3+...+k)+(k+1)<\frac{1}{8}(2k+1)^2+(k+1)$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 19:

Prove the following by using the principle of mathematical induction for all $n \in N$: $n(n + 1)(n + 5)$ is a multiple of 3.

Answer

Let the given statement be P(n), i.e.,

P(n): $n(n + 1)(n + 5)$, which is a multiple of 3.

It can be noted that P(n) is true for $n = 1$ since $1(1 + 1)(1 + 5) = 12$, which is a multiple of 3.

Let P(k) be true for some positive integer k, i.e.,

$k(k + 1)(k + 5)$ is a multiple of 3.

$\therefore k(k + 1)(k + 5) = 3m$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P($k + 1$) is true whenever P(k) is true.

Consider

$$(k+1)\{(k+1)+1\}\{(k+1)+5\}$$
$$=(k+1)(k+2)\{(k+5)+1\}$$
$$=(k+1)(k+2)(k+5)+(k+1)(k+2)$$
$$=\{k(k+1)(k+5)+2(k+1)(k+5)\}+(k+1)(k+2)$$
$$=3m+(k+1)\{2(k+5)+(k+2)\}$$
$$=3m+(k+1)\{2k+10+k+2\}$$
$$=3m+(k+1)(3k+12)$$
$$=3m+3(k+1)(k+4)$$
$$=3\{m+(k+1)(k+4)\}=3\times q, \text{ where } q=\{m+(k+1)(k+4)\} \text{ is some natural number}$$

Therefore, $(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of 3.

Thus, P($k + 1$) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 20:

Prove the following by using the principle of mathematical induction for all $n \in N$: $10^{2n-1} + 1$ is divisible by 11.

Answer

Let the given statement be P(n), i.e., P(n): $10^{2n-1} + 1$ is divisible by 11.

It can be observed that P(n) is true for $n = 1$ since P(1) = $10^{2.1-1} + 1 = 11$, which is divisible by 11.

Let P(k) be true for some positive integer k, i.e.,

$10^{2k-1} + 1$ is divisible by 11.

$\therefore 10^{2k-1} + 1 = 11m$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$10^{2(k+1)-1} + 1$$
$$= 10^{2k+2-1} + 1$$
$$= 10^{2k+1} + 1$$
$$= 10^2\left(10^{2k-1} + 1 - 1\right) + 1$$
$$= 10^2\left(10^{2k-1} + 1\right) - 10^2 + 1$$
$$= 10^2.11m - 100 + 1 \qquad \left[\text{Using }(1)\right]$$
$$= 100 \times 11m - 99$$
$$= 11\left(100m - 9\right)$$
$$= 11r, \text{ where } r = \left(100m - 9\right) \text{ is some natural number}$$

Therefore, $10^{2(k+1)-1} + 1$ is divisible by 11.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 21:

Prove the following by using the principle of mathematical induction for all $n \in N$: $x^{2n} - y^{2n}$ is divisible by $x + y$.

Answer

Let the given statement be P(n), i.e.,

P(n): $x^{2n} - y^{2n}$ is divisible by $x + y$.

It can be observed that P(n) is true for $n = 1$.

This is so because $x^{2 \times 1} - y^{2 \times 1} = x^2 - y^2 = (x + y)(x - y)$ is divisible by $(x + y)$.

Let P(k) be true for some positive integer k, i.e.,

$x^{2k} - y^{2k}$ is divisible by $x + y$.

$\therefore x^{2k} - y^{2k} = m(x + y)$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$x^{2(k+1)} - y^{2(k+1)}$$
$$= x^{2k} \cdot x^2 - y^{2k} \cdot y^2$$
$$= x^2\left(x^{2k} - y^{2k} + y^{2k}\right) - y^{2k} \cdot y^2$$
$$= x^2\left\{m(x+y) + y^{2k}\right\} - y^{2k} \cdot y^2 \qquad [\text{Using (1)}]$$
$$= m(x+y)x^2 + y^{2k} \cdot x^2 - y^{2k} \cdot y^2$$
$$= m(x+y)x^2 + y^{2k}\left(x^2 - y^2\right)$$
$$= m(x+y)x^2 + y^{2k}(x+y)(x-y)$$
$$= (x+y)\left\{mx^2 + y^{2k}(x-y)\right\}, \text{ which is a factor of } (x+y).$$

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 22:

Prove the following by using the principle of mathematical induction for all $n \in N$: $3^{2n+2} - 8n - 9$ is divisible by 8.

Answer

Let the given statement be P(n), i.e.,

P(n): $3^{2n+2} - 8n - 9$ is divisible by 8.

It can be observed that P(n) is true for $n = 1$ since $3^{2 \times 1 + 2} - 8 \times 1 - 9 = 64$, which is divisible by 8.

Let P(k) be true for some positive integer k, i.e.,

$3^{2k+2} - 8k - 9$ is divisible by 8.

$\therefore 3^{2k+2} - 8k - 9 = 8m$; where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$3^{2(k+1)+2}-8(k+1)-9$

$=3^{2k+2}\cdot 3^2-8k-8-9$

$=3^2\left(3^{2k+2}-8k-9+8k+9\right)-8k-17$

$=3^2\left(3^{2k+2}-8k-9\right)+3^2\left(8k+9\right)-8k-17$

$=9.8m+9(8k+9)-8k-17$

$=9.8m+72k+81-8k-17$

$=9.8m+64k+64$

$=8(9m+8k+8)$

$=8r$, where $r=(9m+8k+8)$ is a natural number

Therefore, $3^{2(k+1)+2}-8(k+1)-9$ is divisible by 8.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 23:

Prove the following by using the principle of mathematical induction for all $n \in N$: $41^n - 14^n$ is a multiple of 27.

Answer

Let the given statement be P(n), i.e.,

P(n):$41^n - 14^n$is a multiple of 27.

It can be observed that P(n) is true for n = 1 since $41^1-14^1=27$, which is a multiple of 27.

Let P(k) be true for some positive integer k, i.e.,

$41^k - 14^k$is a multiple of 27

$\therefore 41^k - 14^k = 27m$, where $m \in \mathbf{N}$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$41^{k+1}-14^{k+1}$

$=41^{k}\cdot 41-14^{k}\cdot 14$

$=41\left(41^{k}-14^{k}+14^{k}\right)-14^{k}\cdot 14$

$=41\left(41^{k}-14^{k}\right)+41.14^{k}-14^{k}\cdot 14$

$=41.27m+14^{k}\left(41-14\right)$

$=41.27m+27.14^{k}$

$=27\left(41m-14^{k}\right)$

$=27\times r$, where $r=\left(41m-14^{k}\right)$ is a natural number

Therefore, $41^{k+1}-14^{k+1}$ is a multiple of 27.

Thus, P(k + 1) is true whenever P(k) is true.

Hence, by the principle of mathematical induction, statement P(n) is true for all natural numbers i.e., n.

Question 24:

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$:

$(2n + 7) < (n + 3)^2$

Answer

Let the given statement be P(n), i.e.,

P(n): $(2n + 7) < (n + 3)^2$

It can be observed that P(n) is true for n = 1 since $2.1 + 7 = 9 < (1 + 3)^2 = 16$, which is true.

Let P(k) be true for some positive integer k, i.e.,

$(2k + 7) < (k + 3)^2$ … (1)

We shall now prove that P(k + 1) is true whenever P(k) is true.

Consider

$$\{2(k+1)+7\} = (2k+7)+2$$

$$\therefore \{2(k+1)+7\} = (2k+7)+2 < (k+3)^2+2 \qquad [\text{using } (1)]$$

$$2(k+1)+7 < k^2+6k+9+2$$

$$2(k+1)+7 < k^2+6k+11$$

$$\text{Now, } k^2+6k+11 < k^2+8k+16$$

$$\therefore 2(k+1)+7 < (k+4)^2$$

$$2(k+1)+7 < \{(k+1)+3\}^2$$

Thus, P(*k* + 1) is true whenever P(*k*) is true.

Hence, by the principle of mathematical induction, statement P(*n*) is true for all natural numbers i.e., *n*.