

## Miscellaneous of chapter 13

**Question 1:**

Find the derivative of the following functions from first principle:

(i)  $-x$  (ii)  $(-x)^{-1}$  (iii)  $\sin(x + 1)$

(iv)  $\cos\left(x - \frac{\pi}{8}\right)$

**Answer :**

(i) Let  $f(x) = -x$ . Accordingly,  $f(x+h) = -(x+h)$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(x+h) - (-x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-x - h + x}{h} \\ &= \lim_{h \rightarrow 0} \frac{-h}{h} \\ &= \lim_{h \rightarrow 0} (-1) = -1 \end{aligned}$$

(ii) Let  $f(x) = (-x)^{-1} = \frac{1}{-x} = \frac{-1}{x}$ . Accordingly,  $f(x+h) = \frac{-1}{(x+h)}$

By first principle,

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-1}{x+h} - \left( \frac{-1}{x} \right) \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-1}{x+h} + \frac{1}{x} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-x + (x+h)}{x(x+h)} \right] \\ &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-x + x + h}{x(x+h)} \right] \end{aligned}$$

$$\begin{aligned}
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{h}{x(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{x(x+h)} \\
 &= \frac{1}{x \cdot x} = \frac{1}{x^2}
 \end{aligned}$$

(iii) Let  $f(x) = \sin(x+1)$ . Accordingly,  $f(x+h) = \sin(x+h+1)$

By first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h+1) - \sin(x+1)] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ 2 \cos\left(\frac{x+h+1+x+1}{2}\right) \sin\left(\frac{x+h+1-x-1}{2}\right) \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ 2 \cos\left(\frac{2x+h+2}{2}\right) \sin\left(\frac{h}{2}\right) \right] \\
 &= \lim_{h \rightarrow 0} \left[ \cos\left(\frac{2x+h+2}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right] \\
 &= \lim_{h \rightarrow 0} \cos\left(\frac{2x+h+2}{2}\right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \quad \left[ \text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
 &= \cos\left(\frac{2x+0+2}{2}\right) \cdot 1 \quad \left[ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
 &= \cos(x+1)
 \end{aligned}$$

(iv) Let  $f(x) = \cos\left(x - \frac{\pi}{8}\right)$ . Accordingly,  $f(x+h) = \cos\left(x+h - \frac{\pi}{8}\right)$

By first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \cos\left(x+h - \frac{\pi}{8}\right) - \cos\left(x - \frac{\pi}{8}\right) \right]
 \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ -2 \sin \left( \frac{x+h-\frac{\pi}{8}+x-\frac{\pi}{8}}{2} \right) \sin \left( \frac{x+h-\frac{\pi}{8}-x+\frac{\pi}{8}}{2} \right) \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ -2 \sin \left( \frac{2x+h-\frac{\pi}{4}}{2} \right) \sin \frac{h}{2} \right] \\
&= \lim_{h \rightarrow 0} \left[ -\sin \left( \frac{2x+h-\frac{\pi}{4}}{2} \right) \frac{\sin \left( \frac{h}{2} \right)}{\left( \frac{h}{2} \right)} \right] \\
&= \lim_{h \rightarrow 0} \left[ -\sin \left( \frac{2x+h-\frac{\pi}{4}}{2} \right) \right] \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \left( \frac{h}{2} \right)}{\left( \frac{h}{2} \right)} \quad \left[ \text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
&= -\sin \left( \frac{2x+0-\frac{\pi}{4}}{2} \right) \cdot 1 \\
&= -\sin \left( x - \frac{\pi}{8} \right)
\end{aligned}$$

**Question 2:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $(x + a)$

**Answer :**

Let  $f(x) = x + a$ . Accordingly,  $f(x+h) = x+h+a$

By first principle,

$$\begin{aligned}
f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{x+h+a - x-a}{h} \\
&= \lim_{h \rightarrow 0} \left( \frac{h}{h} \right) \\
&= \lim_{h \rightarrow 0} (1) \\
&= 1
\end{aligned}$$

**Question 3:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

$$(px+q)\left(\frac{r}{x}+s\right)$$

fixed non-zero constants and  $m$  and  $n$  are integers):

**Answer :**

$$\text{Let } f(x) = (px + q) \left( \frac{r}{x} + s \right)$$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (px + q) \left( \frac{r}{x} + s \right)' + \left( \frac{r}{x} + s \right) (px + q)' \\ &= (px + q) (rx^{-1} + s)' + \left( \frac{r}{x} + s \right) (p) \\ &= (px + q) (-rx^{-2}) + \left( \frac{r}{x} + s \right) p \\ &= (px + q) \left( \frac{-r}{x^2} \right) + \left( \frac{r}{x} + s \right) p \\ &= \frac{-pr}{x} - \frac{qr}{x^2} + \frac{pr}{x} + ps \\ &= ps - \frac{qr}{x^2} \end{aligned}$$

**Question 4:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $(ax + b)(cx + d)^2$

**Answer :**

$$\text{Let } f(x) = (ax + b)(cx + d)^2$$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (ax + b) \frac{d}{dx} (cx + d)^2 + (cx + d)^2 \frac{d}{dx} (ax + b) \\ &= (ax + b) \frac{d}{dx} (c^2x^2 + 2cdx + d^2) + (cx + d)^2 \frac{d}{dx} (ax + b) \\ &= (ax + b) \left[ \frac{d}{dx} (c^2x^2) + \frac{d}{dx} (2cdx) + \frac{d}{dx} d^2 \right] + (cx + d)^2 \left[ \frac{d}{dx} ax + \frac{d}{dx} b \right] \\ &= (ax + b) (2c^2x + 2cd) + (cx + d)^2 a \\ &= 2c(ax + b)(cx + d) + a(cx + d)^2 \end{aligned}$$

**Question 5:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{ax + b}{cx + d}$

**Answer :**

$$\text{Let } f(x) = \frac{ax + b}{cx + d}$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(cx+d)\frac{d}{dx}(ax+b) - (ax+b)\frac{d}{dx}(cx+d)}{(cx+d)^2} \\ &= \frac{(cx+d)(a) - (ax+b)(c)}{(cx+d)^2} \\ &= \frac{acx + ad - acx - bc}{(cx+d)^2} \\ &= \frac{ad - bc}{(cx+d)^2} \end{aligned}$$

**Question 6:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):

$$1 + \frac{1}{x}$$

$$\frac{x}{1 - \frac{1}{x}}$$

**Answer :**

$$\text{Let } f(x) = \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}, \text{ where } x \neq 0$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x-1)\frac{d}{dx}(x+1) - (x+1)\frac{d}{dx}(x-1)}{(x-1)^2}, x \neq 0, 1 \\ &= \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}, x \neq 0, 1 \\ &= \frac{x-1-x-1}{(x-1)^2}, x \neq 0, 1 \\ &= \frac{-2}{(x-1)^2}, x \neq 0, 1 \end{aligned}$$

**Question 7:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):

$$\frac{1}{ax^2 + bx + c}$$

**Answer :**

$$\text{Let } f(x) = \frac{1}{ax^2 + bx + c}$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(ax^2 + bx + c) \frac{d}{dx}(1) - \frac{d}{dx}(ax^2 + bx + c)}{(ax^2 + bx + c)^2} \\ &= \frac{(ax^2 + bx + c)(0) - (2ax + b)}{(ax^2 + bx + c)^2} \\ &= \frac{-(2ax + b)}{(ax^2 + bx + c)^2} \end{aligned}$$

**Question 8:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{ax + b}{px^2 + qx + r}$

**Answer :**

$$\text{Let } f(x) = \frac{ax + b}{px^2 + qx + r}$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(px^2 + qx + r) \frac{d}{dx}(ax + b) - (ax + b) \frac{d}{dx}(px^2 + qx + r)}{(px^2 + qx + r)^2} \\ &= \frac{(px^2 + qx + r)(a) - (ax + b)(2px + q)}{(px^2 + qx + r)^2} \\ &= \frac{apx^2 + aqx + ar - 2apx^2 - aqx - 2bpx - bq}{(px^2 + qx + r)^2} \\ &= \frac{-apx^2 - 2bpx + ar - bq}{(px^2 + qx + r)^2} \end{aligned}$$

**Question 9:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{px^2 + qx + r}{ax + b}$

**Answer :**

$$\text{Let } f(x) = \frac{px^2 + qx + r}{ax + b}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(ax+b) \frac{d}{dx}(px^2+qx+r) - (px^2+qx+r) \frac{d}{dx}(ax+b)}{(ax+b)^2} \\
 &= \frac{(ax+b)(2px+q) - (px^2+qx+r)(a)}{(ax+b)^2} \\
 &= \frac{2apx^2 + aqx + 2bpx + bq - apx^2 - aqx - ar}{(ax+b)^2} \\
 &= \frac{apx^2 + 2bpx + bq - ar}{(ax+b)^2}
 \end{aligned}$$

**Question 10:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{a}{x^4} - \frac{b}{x^2} + \cos x$

**Answer :**

$$\begin{aligned}
 \text{Let } f(x) &= \frac{a}{x^4} - \frac{b}{x^2} + \cos x \\
 f'(x) &= \frac{d}{dx} \left( \frac{a}{x^4} \right) - \frac{d}{dx} \left( \frac{b}{x^2} \right) + \frac{d}{dx} (\cos x) \\
 &= a \frac{d}{dx} (x^{-4}) - b \frac{d}{dx} (x^{-2}) + \frac{d}{dx} (\cos x) \\
 &= a(-4x^{-5}) - b(-2x^{-3}) + (-\sin x) \quad \left[ \frac{d}{dx} (x^n) = nx^{n-1} \text{ and } \frac{d}{dx} (\cos x) = -\sin x \right] \\
 &= \frac{-4a}{x^5} + \frac{2b}{x^3} - \sin x
 \end{aligned}$$

**Question 11:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $4\sqrt{x} - 2$

**Answer :**

$$\begin{aligned}
 \text{Let } f(x) &= 4\sqrt{x} - 2 \\
 f'(x) &= \frac{d}{dx} (4\sqrt{x} - 2) = \frac{d}{dx} (4\sqrt{x}) - \frac{d}{dx} (2) \\
 &= 4 \frac{d}{dx} \left( x^{\frac{1}{2}} \right) - 0 = 4 \left( \frac{1}{2} x^{\frac{1}{2}-1} \right) \\
 &= \left( 2x^{-\frac{1}{2}} \right) = \frac{2}{\sqrt{x}}
 \end{aligned}$$

**Question 12:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $(ax + b)^n$

**Answer :**

Let  $f(x) = (ax + b)^n$ . Accordingly,  $f(x + h) = \{a(x + h) + b\}^n = (ax + ah + b)^n$

By first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(ax + ah + b)^n - (ax + b)^n}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(ax + b)^n \left(1 + \frac{ah}{ax + b}\right)^n - (ax + b)^n}{h} \\
 &= (ax + b)^n \lim_{h \rightarrow 0} \frac{\left(1 + \frac{ah}{ax + b}\right)^n - 1}{h} \\
 &= (ax + b)^n \lim_{h \rightarrow 0} \frac{1}{n} \left[ \left\{ 1 + n \left(\frac{ah}{ax + b}\right) + \frac{n(n-1)}{2} \left(\frac{ah}{ax + b}\right)^2 + \dots \right\} - 1 \right] \\
 &\qquad\qquad\qquad \text{(Using binomial theorem)} \\
 &= (ax + b)^n \lim_{h \rightarrow 0} \frac{1}{h} \left[ n \left(\frac{ah}{ax + b}\right) + \frac{n(n-1)a^2h^2}{2(ax + b)^2} + \dots \text{(Terms containing higher degrees of } h) \right] \\
 &= (ax + b)^n \lim_{h \rightarrow 0} \left[ \frac{na}{(ax + b)} + \frac{n(n-1)a^2h}{2(ax + b)^2} + \dots \right] \\
 &= (ax + b)^n \left[ \frac{na}{(ax + b)} + 0 \right] \\
 &= na \frac{(ax + b)^n}{(ax + b)} \\
 &= na(ax + b)^{n-1}
 \end{aligned}$$

**Question 13:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $(ax + b)^n (cx + d)^m$

**Answer :**

Let  $f(x) = (ax + b)^n (cx + d)^m$

By Leibnitz product rule,



$$f'(x) = (ax+b)^n \frac{d}{dx}(cx+d)^m + (cx+d)^m \frac{d}{dx}(ax+b)^n \quad \dots(1)$$

$$\text{Now, let } f_1(x) = (cx+d)^m$$

$$f_1(x+h) = (cx+ch+d)^m$$

$$f_1'(x) = \lim_{h \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h}$$

$$= \lim_{h \rightarrow 0} \frac{(cx+ch+d)^m - (cx+d)^m}{h}$$

$$= (cx+d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[ \left( 1 + \frac{ch}{cx+d} \right)^m - 1 \right]$$

$$= (cx+d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[ \left( 1 + \frac{mch}{(cx+d)} + \frac{m(m-1)}{2} \frac{(c^2h^2)}{(cx+d)^2} + \dots \right) - 1 \right]$$

$$= (cx+d)^m \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{mch}{(cx+d)} + \frac{m(m-1)c^2h^2}{2(cx+d)^2} + \dots (\text{Terms containing higher degrees of } h) \right]$$

$$= (cx+d)^m \lim_{h \rightarrow 0} \left[ \frac{mc}{(cx+d)} + \frac{m(m-1)c^2h}{2(cx+d)^2} + \dots \right]$$

$$= (cx+d)^m \left[ \frac{mc}{cx+d} + 0 \right]$$

$$= \frac{mc(cx+d)^m}{(cx+d)}$$

$$= mc(cx+d)^{m-1}$$

$$\frac{d}{dx}(cx+d)^m = mc(cx+d)^{m-1} \quad \dots(2)$$

$$\text{Similarly, } \frac{d}{dx}(ax+b)^n = na(ax+b)^{n-1} \quad \dots(3)$$

Therefore, from (1), (2), and (3), we obtain

$$\begin{aligned} f'(x) &= (ax+b)^n \{ mc(cx+d)^{m-1} \} + (cx+d)^m \{ na(ax+b)^{n-1} \} \\ &= (ax+b)^{n-1} (cx+d)^{m-1} [ mc(ax+b) + na(cx+d) ] \end{aligned}$$

#### Question 14:

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $\sin(x+a)$

**Answer :**

$$\text{Let } f(x) = \sin(x+a)$$

$$f(x+h) = \sin(x+h+a)$$

By first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h+a) - \sin(x+a)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ 2 \cos\left(\frac{x+h+a+x+a}{2}\right) \sin\left(\frac{x+h+a-x-a}{2}\right) \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ 2 \cos\left(\frac{2x+2a+h}{2}\right) \sin\left(\frac{h}{2}\right) \right] \\
 &= \lim_{h \rightarrow 0} \left[ \cos\left(\frac{2x+2a+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \right] \\
 &= \lim_{h \rightarrow 0} \cos\left(\frac{2x+2a+h}{2}\right) \lim_{\frac{h}{2} \rightarrow 0} \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \quad \left[ \text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
 &= \cos\left(\frac{2x+2a}{2}\right) \times 1 \quad \left[ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
 &= \cos(x+a)
 \end{aligned}$$

**Question 15:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $\operatorname{cosec} x \cot x$

**Answer :**

$$\text{Let } f(x) = \operatorname{cosec} x \cot x$$

By Leibnitz product rule,

$$f'(x) = \operatorname{cosec} x (\cot x)' + \cot x (\operatorname{cosec} x)' \quad \dots(1)$$

$$\text{Let } f_1(x) = \cot x. \text{ Accordingly, } f_1(x+h) = \cot(x+h)$$

By first principle,

$$\begin{aligned}
 f_1'(x) &= \lim_{h \rightarrow 0} \frac{f_1(x+h) - f_1(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cot(x+h) - \cot x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left( \frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin x \cos(x+h) - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right] \\
 &= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(-h)}{\sin(x+h)} \right] \\
 &= \frac{-1}{\sin x} \cdot \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left( \lim_{h \rightarrow 0} \frac{1}{\sin(x+h)} \right) \\
 &= \frac{-1}{\sin x} \cdot 1 \cdot \left( \frac{1}{\sin(x+0)} \right) \\
 &= \frac{-1}{\sin^2 x} \\
 &= -\operatorname{cosec}^2 x
 \end{aligned}$$

$$\therefore (\cot x)' = -\operatorname{cosec}^2 x \quad \dots(2)$$

Now, let  $f_2(x) = \operatorname{cosec} x$ . Accordingly,  $f_2(x+h) = \operatorname{cosec}(x+h)$

By first principle,

$$\begin{aligned}
 f_2'(x) &= \lim_{h \rightarrow 0} \frac{f_2(x+h) - f_2(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x]
 \end{aligned}$$

$$\begin{aligned}
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin x - \sin(x+h)}{\sin x \sin(x+h)} \right] \\
&= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{2 \cos\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)} \right] \\
&= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(\frac{-h}{2}\right)}{\sin(x+h)} \right] \\
&= \frac{1}{\sin x} \cdot \lim_{h \rightarrow 0} \left[ \frac{-\sin\left(\frac{h}{2}\right) \cos\left(\frac{2x+h}{2}\right)}{\left(\frac{h}{2}\right) \sin(x+h)} \right] \\
&= \frac{-1}{\sin x} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)} \\
&= \frac{-1}{\sin x} \cdot 1 \cdot \frac{\cos\left(\frac{2x+0}{2}\right)}{\sin(x+0)} \\
&= \frac{-1}{\sin x} \cdot \frac{\cos x}{\sin x} \\
&= -\operatorname{cosec} x \cdot \cot x
\end{aligned}$$

$$\therefore (\operatorname{cosec} x)' = -\operatorname{cosec} x \cdot \cot x \quad \dots(3)$$

From (1), (2), and (3), we obtain

$$\begin{aligned}
f'(x) &= \operatorname{cosec} x (-\operatorname{cosec}^2 x) + \cot x (-\operatorname{cosec} x \cot x) \\
&= -\operatorname{cosec}^3 x - \cot^2 x \operatorname{cosec} x
\end{aligned}$$

**Question 16:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{\cos x}{1 + \sin x}$

**Answer :**

$$\text{Let } f(x) = \frac{\cos x}{1 + \sin x}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(1 + \sin x) \frac{d}{dx}(\cos x) - (\cos x) \frac{d}{dx}(1 + \sin x)}{(1 + \sin x)^2} \\
 &= \frac{(1 + \sin x)(-\sin x) - (\cos x)(\cos x)}{(1 + \sin x)^2} \\
 &= \frac{-\sin x - \sin^2 x - \cos^2 x}{(1 + \sin x)^2} \\
 &= \frac{-\sin x - (\sin^2 x + \cos^2 x)}{(1 + \sin x)^2} \\
 &= \frac{-\sin x - 1}{(1 + \sin x)^2} \\
 &= \frac{-(1 + \sin x)}{(1 + \sin x)^2} \\
 &= \frac{-1}{(1 + \sin x)}
 \end{aligned}$$

**Question 17:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{\sin x + \cos x}{\sin x - \cos x}$

**Answer :**

$$f(x) = \frac{\sin x + \cos x}{\sin x - \cos x}$$

Let

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(\sin x - \cos x) \frac{d}{dx}(\sin x + \cos x) - (\sin x + \cos x) \frac{d}{dx}(\sin x - \cos x)}{(\sin x - \cos x)^2} \\
 &= \frac{(\sin x - \cos x)(\cos x - \sin x) - (\sin x + \cos x)(\cos x + \sin x)}{(\sin x - \cos x)^2} \\
 &= \frac{-(\sin x - \cos x)^2 - (\sin x + \cos x)^2}{(\sin x - \cos x)^2} \\
 &= \frac{-[\sin^2 x + \cos^2 x - 2 \sin x \cos x + \sin^2 x + \cos^2 x + 2 \sin x \cos x]}{(\sin x - \cos x)^2} \\
 &= \frac{-[1 + 1]}{(\sin x - \cos x)^2} \\
 &= \frac{-2}{(\sin x - \cos x)^2}
 \end{aligned}$$

**Question 18:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{\sec x - 1}{\sec x + 1}$

**Answer :**

$$\text{Let } f(x) = \frac{\sec x - 1}{\sec x + 1}$$

$$f(x) = \frac{\frac{1}{\cos x} - 1}{\frac{1}{\cos x} + 1} = \frac{1 - \cos x}{1 + \cos x}$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(1 + \cos x) \frac{d}{dx}(1 - \cos x) - (1 - \cos x) \frac{d}{dx}(1 + \cos x)}{(1 + \cos x)^2} \\ &= \frac{(1 + \cos x)(\sin x) - (1 - \cos x)(-\sin x)}{(1 + \cos x)^2} \\ &= \frac{\sin x + \cos x \sin x + \sin x - \sin x \cos x}{(1 + \cos x)^2} \\ &= \frac{2 \sin x}{(1 + \cos x)^2} \\ &= \frac{2 \sin x}{\left(1 + \frac{1}{\sec x}\right)^2} = \frac{2 \sin x}{\frac{(\sec x + 1)^2}{\sec^2 x}} \\ &= \frac{2 \sin x \sec^2 x}{(\sec x + 1)^2} \\ &= \frac{2 \sin x \sec x}{\cos x (\sec x + 1)^2} \\ &= \frac{2 \sec x \tan x}{(\sec x + 1)^2} \end{aligned}$$

**Question 19:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $\sin^n x$

**Answer :**

Let  $y = \sin^n x$ .

Accordingly, for  $n = 1$ ,  $y = \sin x$ .

$$\therefore \frac{dy}{dx} = \cos x, \text{ i.e., } \frac{d}{dx} \sin x = \cos x$$

For  $n = 2$ ,  $y = \sin^2 x$ .

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx}(\sin x \sin x) \\ &= (\sin x)' \sin x + \sin x (\sin x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin x + \sin x \cos x \\ &= 2 \sin x \cos x && \dots(1) \end{aligned}$$

For  $n = 3$ ,  $y = \sin^3 x$ .

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{d}{dx}(\sin x \sin^2 x) \\ &= (\sin x)' \sin^2 x + \sin x (\sin^2 x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin^2 x + \sin x (2 \sin x \cos x) && \text{[Using (1)]} \\ &= \cos x \sin^2 x + 2 \sin^2 x \cos x \\ &= 3 \sin^2 x \cos x \end{aligned}$$

$$\text{We assert that } \frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$$

Let our assertion be true for  $n = k$ .

$$\text{i.e., } \frac{d}{dx}(\sin^k x) = k \sin^{(k-1)} x \cos x \quad \dots(2)$$

Consider

$$\begin{aligned} \frac{d}{dx}(\sin^{k+1} x) &= \frac{d}{dx}(\sin x \sin^k x) \\ &= (\sin x)' \sin^k x + \sin x (\sin^k x)' && \text{[By Leibnitz product rule]} \\ &= \cos x \sin^k x + \sin x (k \sin^{(k-1)} x \cos x) && \text{[Using (2)]} \\ &= \cos x \sin^k x + k \sin^k x \cos x \\ &= (k+1) \sin^k x \cos x \end{aligned}$$

Thus, our assertion is true for  $n = k + 1$ .

$$\text{Hence, by mathematical induction, } \frac{d}{dx}(\sin^n x) = n \sin^{(n-1)} x \cos x$$

#### Question 20:

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{a + b \sin x}{c + d \cos x}$

**Answer :**

$$\text{Let } f(x) = \frac{a + b \sin x}{c + d \cos x}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(c+d \cos x) \frac{d}{dx}(a+b \sin x) - (a+b \sin x) \frac{d}{dx}(c+d \cos x)}{(c+d \cos x)^2} \\
 &= \frac{(c+d \cos x)(b \cos x) - (a+b \sin x)(-d \sin x)}{(c+d \cos x)^2} \\
 &= \frac{cb \cos x + bd \cos^2 x + ad \sin x + bd \sin^2 x}{(c+d \cos x)^2} \\
 &= \frac{bc \cos x + ad \sin x + bd(\cos^2 x + \sin^2 x)}{(c+d \cos x)^2} \\
 &= \frac{bc \cos x + ad \sin x + bd}{(c+d \cos x)^2}
 \end{aligned}$$

**Question 21:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{\sin(x+a)}{\cos x}$

**Answer :**

$$\text{Let } f(x) = \frac{\sin(x+a)}{\cos x}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{\cos x \frac{d}{dx}[\sin(x+a)] - \sin(x+a) \frac{d}{dx} \cos x}{\cos^2 x} \\
 f'(x) &= \frac{\cos x \frac{d}{dx}[\sin(x+a)] - \sin(x+a)(-\sin x)}{\cos^2 x} \quad \dots (i)
 \end{aligned}$$

Let  $g(x) = \sin(x+a)$ . Accordingly,  $g(x+h) = \sin(x+h+a)$

By first principle,



$$\begin{aligned}
g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h+a) - \sin(x+a)] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ 2 \cos\left(\frac{x+h+a+x+a}{2}\right) \sin\left(\frac{x+h+a-x-a}{2}\right) \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ 2 \cos\left(\frac{2x+2a+h}{2}\right) \sin\left(\frac{h}{2}\right) \right] \\
&= \lim_{h \rightarrow 0} \left[ \cos\left(\frac{2x+2a+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \right] \\
&= \lim_{h \rightarrow 0} \cos\left(\frac{2x+2a+h}{2}\right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \right\} \quad \left[ \text{As } h \rightarrow 0 \Rightarrow \frac{h}{2} \rightarrow 0 \right] \\
&= \left( \cos \frac{2x+2a}{2} \right) \times 1 \quad \left[ \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
&= \cos(x+a) \quad \dots \text{ (ii)}
\end{aligned}$$

From (i) and (ii), we obtain

$$\begin{aligned}
f'(x) &= \frac{\cos x \cdot \cos(x+a) + \sin x \sin(x+a)}{\cos^2 x} \\
&= \frac{\cos(x+a-x)}{\cos^2 x} \\
&= \frac{\cos a}{\cos^2 x}
\end{aligned}$$

**Question 22:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $x^4 (5 \sin x - 3 \cos x)$

**Answer :**

$$\text{Let } f(x) = x^4 (5 \sin x - 3 \cos x)$$

By product rule,

$$\begin{aligned}
f'(x) &= x^4 \frac{d}{dx} (5 \sin x - 3 \cos x) + (5 \sin x - 3 \cos x) \frac{d}{dx} (x^4) \\
&= x^4 \left[ 5 \frac{d}{dx} (\sin x) - 3 \frac{d}{dx} (\cos x) \right] + (5 \sin x - 3 \cos x) \frac{d}{dx} (x^4) \\
&= x^4 [5 \cos x - 3(-\sin x)] + (5 \sin x - 3 \cos x)(4x^3) \\
&= x^3 [5x \cos x + 3x \sin x + 20 \sin x - 12 \cos x]
\end{aligned}$$

**Question 23:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $(x^2 + 1) \cos x$

**Answer :**

$$\text{Let } f(x) = (x^2 + 1) \cos x$$

By product rule,

$$\begin{aligned} f'(x) &= (x^2 + 1) \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(x^2 + 1) \\ &= (x^2 + 1)(-\sin x) + \cos x(2x) \\ &= -x^2 \sin x - \sin x + 2x \cos x \end{aligned}$$

**Question 24:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $(ax^2 + \sin x)(p + q \cos x)$

**Answer :**

$$\text{Let } f(x) = (ax^2 + \sin x)(p + q \cos x)$$

By product rule,

$$\begin{aligned} f'(x) &= (ax^2 + \sin x) \frac{d}{dx}(p + q \cos x) + (p + q \cos x) \frac{d}{dx}(ax^2 + \sin x) \\ &= (ax^2 + \sin x)(-q \sin x) + (p + q \cos x)(2ax + \cos x) \\ &= -q \sin x(ax^2 + \sin x) + (p + q \cos x)(2ax + \cos x) \end{aligned}$$

**Question 25:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $(x + \cos x)(x - \tan x)$

**Answer :**

$$\text{Let } f(x) = (x + \cos x)(x - \tan x)$$

By product rule,

$$\begin{aligned} f'(x) &= (x + \cos x) \frac{d}{dx}(x - \tan x) + (x - \tan x) \frac{d}{dx}(x + \cos x) \\ &= (x + \cos x) \left[ \frac{d}{dx}(x) - \frac{d}{dx}(\tan x) \right] + (x - \tan x)(1 - \sin x) \\ &= (x + \cos x) \left[ 1 - \frac{d}{dx} \tan x \right] + (x - \tan x)(1 - \sin x) \quad \dots (i) \end{aligned}$$

$$\text{Let } g(x) = \tan x. \text{ Accordingly, } g(x+h) = \tan(x+h)$$

By first principle,

$$\begin{aligned}
 g'(x) &= \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} \\
 &= \lim_{h \rightarrow 0} \left( \frac{\tan(x+h) - \tan x}{h} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h-x)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin h}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left( \lim_{h \rightarrow 0} \frac{1}{\cos(x+h)} \right) \\
 &= \frac{1}{\cos x} \cdot 1 \cdot \frac{1}{\cos(x+0)} \\
 &= \frac{1}{\cos^2 x} \\
 &= \sec^2 x \quad \dots \text{(ii)}
 \end{aligned}$$

Therefore, from (i) and (ii), we obtain

$$\begin{aligned}
 f'(x) &= (x + \cos x)(1 - \sec^2 x) + (x - \tan x)(1 - \sin x) \\
 &= (x + \cos x)(-\tan^2 x) + (x - \tan x)(1 - \sin x) \\
 &= -\tan^2 x(x + \cos x) + (x - \tan x)(1 - \sin x)
 \end{aligned}$$

**Question 26:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{4x + 5 \sin x}{3x + 7 \cos x}$

**Answer :**

$$\text{Let } f(x) = \frac{4x + 5 \sin x}{3x + 7 \cos x}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \frac{(3x+7\cos x)\frac{d}{dx}(4x+5\sin x) - (4x+5\sin x)\frac{d}{dx}(3x+7\cos x)}{(3x+7\cos x)^2} \\
 &= \frac{(3x+7\cos x)\left[4\frac{d}{dx}(x)+5\frac{d}{dx}(\sin x)\right] - (4x+5\sin x)\left[3\frac{d}{dx}x+7\frac{d}{dx}\cos x\right]}{(3x+7\cos x)^2} \\
 &= \frac{(3x+7\cos x)(4+5\cos x) - (4x+5\sin x)(3-7\sin x)}{(3x+7\cos x)^2} \\
 &= \frac{12x+15x\cos x+28\cos x+35\cos^2 x - 12x+28x\sin x - 15\sin x+35\sin^2 x}{(3x+7\cos x)^2} \\
 &= \frac{15x\cos x+28\cos x+28x\sin x - 15\sin x+35(\cos^2 x+\sin^2 x)}{(3x+7\cos x)^2} \\
 &= \frac{35+15x\cos x+28\cos x+28x\sin x - 15\sin x}{(3x+7\cos x)^2}
 \end{aligned}$$

**Question 27:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):

$$\frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$$

**Answer :**

$$\text{Let } f(x) = \frac{x^2 \cos\left(\frac{\pi}{4}\right)}{\sin x}$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \cos\frac{\pi}{4} \cdot \left[ \frac{\sin x \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(\sin x)}{\sin^2 x} \right] \\
 &= \cos\frac{\pi}{4} \cdot \left[ \frac{\sin x \cdot 2x - x^2 \cos x}{\sin^2 x} \right] \\
 &= \frac{x \cos\frac{\pi}{4} [2\sin x - x \cos x]}{\sin^2 x}
 \end{aligned}$$

**Question 28:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{x}{1+\tan x}$

**Answer :**

$$\text{Let } f(x) = \frac{x}{1 + \tan x}$$

$$f'(x) = \frac{(1 + \tan x) \frac{d}{dx}(x) - x \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2}$$

$$f'(x) = \frac{(1 + \tan x) - x \cdot \frac{d}{dx}(1 + \tan x)}{(1 + \tan x)^2} \quad \dots \text{ (i)}$$

Let  $g(x) = 1 + \tan x$ . Accordingly,  $g(x+h) = 1 + \tan(x+h)$ .

By first principle,

$$g'(x) = \lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \rightarrow 0} \left[ \frac{1 + \tan(x+h) - 1 - \tan x}{h} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h-x)}{\cos(x+h)\cos x} \right]$$

$$= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin h}{\cos(x+h)\cos x} \right]$$

$$= \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left( \lim_{h \rightarrow 0} \frac{1}{\cos(x+h)\cos x} \right)$$

$$= 1 \times \frac{1}{\cos^2 x} = \sec^2 x$$

$$\Rightarrow \frac{d}{dx}(1 + \tan x) = \sec^2 x \quad \dots \text{ (ii)}$$

From (i) and (ii), we obtain

$$f'(x) = \frac{1 + \tan x - x \sec^2 x}{(1 + \tan x)^2}$$

#### Question 29:

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are fixed non-zero constants and  $m$  and  $n$  are integers):  $(x + \sec x)(x - \tan x)$

**Answer :**

$$\text{Let } f(x) = (x + \sec x)(x - \tan x)$$

By product rule,

$$\begin{aligned}
 f'(x) &= (x + \sec x) \frac{d}{dx}(x - \tan x) + (x - \tan x) \frac{d}{dx}(x + \sec x) \\
 &= (x + \sec x) \left[ \frac{d}{dx}(x) - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[ \frac{d}{dx}(x) + \frac{d}{dx} \sec x \right] \\
 &= (x + \sec x) \left[ 1 - \frac{d}{dx} \tan x \right] + (x - \tan x) \left[ 1 + \frac{d}{dx} \sec x \right] \quad \dots (i)
 \end{aligned}$$

Let  $f_1(x) = \tan x$ ,  $f_2(x) = \sec x$

Accordingly,  $f_1(x+h) = \tan(x+h)$  and  $f_2(x+h) = \sec(x+h)$

$$\begin{aligned}
 f_1'(x) &= \lim_{h \rightarrow 0} \left( \frac{f_1(x+h) - f_1(x)}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left( \frac{\tan(x+h) - \tan x}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left[ \frac{\tan(x+h) - \tan x}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h)\cos x - \sin x \cos(x+h)}{\cos(x+h)\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h-x)}{\cos(x+h)\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin h}{\cos(x+h)\cos x} \right] \\
 &= \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left( \lim_{h \rightarrow 0} \frac{1}{\cos(x+h)\cos x} \right) \\
 &= 1 \times \frac{1}{\cos^2 x} = \sec^2 x \\
 \Rightarrow \frac{d}{dx} \tan x &= \sec^2 x \quad \dots (ii)
 \end{aligned}$$

$$\begin{aligned}
 f_2'(x) &= \lim_{h \rightarrow 0} \left( \frac{f_2(x+h) - f_2(x)}{h} \right) \\
 &= \lim_{h \rightarrow 0} \left( \frac{\sec(x+h) - \sec x}{h} \right) \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\cos x - \cos(x+h)}{\cos(x+h)\cos x} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2 \sin\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2 \sin\left(\frac{2x+h}{2}\right) \cdot \sin\left(\frac{-h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \left[ \frac{\sin\left(\frac{2x+h}{2}\right) \left\{ \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\cos(x+h)} \right] \\
 &= \sec x \cdot \frac{\left\{ \lim_{h \rightarrow 0} \sin\left(\frac{2x+h}{2}\right) \right\} \left\{ \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right\}}{\lim_{h \rightarrow 0} \cos(x+h)} \\
 &= \sec x \cdot \frac{\sin x \cdot 1}{\cos x} \\
 \Rightarrow \frac{d}{dx} \sec x &= \sec x \tan x \quad \dots \text{.. (iii)}
 \end{aligned}$$

From (i), (ii), and (iii), we obtain

$$f'(x) = (x + \sec x)(1 - \sec^2 x) + (x - \tan x)(1 + \sec x \tan x)$$

**Question 30:**

Find the derivative of the following functions (it is to be understood that  $a, b, c, d, p, q, r$  and  $s$  are

fixed non-zero constants and  $m$  and  $n$  are integers):  $\frac{x}{\sin^n x}$

**Answer :**

$$\text{Let } f(x) = \frac{x}{\sin^n x}$$

By quotient rule,

$$f'(x) = \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x}$$

It can be easily shown that  $\frac{d}{dx} \sin^n x = n \sin^{n-1} x \cos x$

Therefore,

$$\begin{aligned} f'(x) &= \frac{\sin^n x \frac{d}{dx} x - x \frac{d}{dx} \sin^n x}{\sin^{2n} x} \\ &= \frac{\sin^n x \cdot 1 - x(n \sin^{n-1} x \cos x)}{\sin^{2n} x} \\ &= \frac{\sin^{n-1} x (\sin x - nx \cos x)}{\sin^{2n} x} \\ &= \frac{\sin x - nx \cos x}{\sin^{n+1} x} \end{aligned}$$



## EXCERSIE 13.1 SOULTIONS

**Question 1:**

Evaluate the Given limit:  $\lim_{x \rightarrow 3} x + 3$

**Answer :**

$$\lim_{x \rightarrow 3} x + 3 = 3 + 3 = 6$$

**Question 2:**

Evaluate the Given limit:  $\lim_{x \rightarrow \pi} \left( x - \frac{22}{7} \right)$

**Answer :**

$$\lim_{x \rightarrow \pi} \left( x - \frac{22}{7} \right) = \left( \pi - \frac{22}{7} \right)$$

**Question 3:**

Evaluate the Given limit:  $\lim_{r \rightarrow 1} \pi r^2$

**Answer :**

$$\lim_{r \rightarrow 1} \pi r^2 = \pi (1)^2 = \pi$$

**Question 4:**

Evaluate the Given limit:  $\lim_{x \rightarrow 4} \frac{4x+3}{x-2}$

**Answer :**

$$\lim_{x \rightarrow 4} \frac{4x+3}{x-2} = \frac{4(4)+3}{4-2} = \frac{16+3}{2} = \frac{19}{2}$$

**Question 5:**

Evaluate the Given limit:  $\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x-1}$

**Answer :**

$$\lim_{x \rightarrow -1} \frac{x^{10} + x^5 + 1}{x-1} = \frac{(-1)^{10} + (-1)^5 + 1}{-1-1} = \frac{1-1+1}{-2} = -\frac{1}{2}$$

**Question 6:**

Evaluate the Given limit:  $\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$

**Answer :**

$$\lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x}$$

Put  $x + 1 = y$  so that  $y \rightarrow 1$  as  $x \rightarrow 0$ .

$$\begin{aligned} \text{Accordingly, } \lim_{x \rightarrow 0} \frac{(x+1)^5 - 1}{x} &= \lim_{y \rightarrow 1} \frac{y^5 - 1}{y - 1} \\ &= \lim_{y \rightarrow 1} \frac{y^5 - 1^5}{y - 1} \\ &= 5 \cdot 1^{5-1} \\ &= 5 \end{aligned}$$

$$\left[ \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right]$$

$$\therefore \lim_{x \rightarrow 0} \frac{(x+5)^5 - 1}{x} = 5$$

**Question 7:**

$$\lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4}$$

Evaluate the Given limit.

**Answer :**

At  $x = 2$ , the value of the given rational function takes the form  $\frac{0}{0}$ .

$$\begin{aligned} \therefore \lim_{x \rightarrow 2} \frac{3x^2 - x - 10}{x^2 - 4} &= \lim_{x \rightarrow 2} \frac{(x-2)(3x+5)}{(x-2)(x+2)} \\ &= \lim_{x \rightarrow 2} \frac{3x+5}{x+2} \\ &= \frac{3(2)+5}{2+2} \\ &= \frac{11}{4} \end{aligned}$$

**Question 8:**

$$\lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3}$$

Evaluate the Given limit.

**Answer :**

At  $x = 2$ , the value of the given rational function takes the form  $\frac{0}{0}$ .

$$\begin{aligned}
 \therefore \lim_{x \rightarrow 3} \frac{x^4 - 81}{2x^2 - 5x - 3} &= \lim_{x \rightarrow 3} \frac{(x-3)(x+3)(x^2+9)}{(x-3)(2x+1)} \\
 &= \lim_{x \rightarrow 3} \frac{(x+3)(x^2+9)}{2x+1} \\
 &= \frac{(3+3)(3^2+9)}{2(3)+1} \\
 &= \frac{6 \times 18}{7} \\
 &= \frac{108}{7}
 \end{aligned}$$

**Question 9:**

Evaluate the Given limit:  $\lim_{x \rightarrow 0} \frac{ax+b}{cx+1}$

**Answer :**

$$\lim_{x \rightarrow 0} \frac{ax+b}{cx+1} = \frac{a(0)+b}{c(0)+1} = b$$

**Question 10:**

Evaluate the Given limit:  $\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$

**Answer :**

$$\lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1}$$

At  $z = 1$ , the value of the given function takes the form  $\frac{0}{0}$ .

Put  $z^{\frac{1}{6}} = x$  so that  $z \rightarrow 1$  as  $x \rightarrow 1$ .

$$\begin{aligned}
 \text{Accordingly, } \lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \\
 &= \lim_{x \rightarrow 1} \frac{x^2 - 1^2}{x - 1} \\
 &= 2 \cdot 1^{2-1} \quad \left[ \lim_{x \rightarrow a} \frac{x^n - a^n}{x - a} = na^{n-1} \right] \\
 &= 2
 \end{aligned}$$

$$\therefore \lim_{z \rightarrow 1} \frac{z^{\frac{1}{3}} - 1}{z^{\frac{1}{6}} - 1} = 2$$

**Question 11:**

Evaluate the Given limit:  $\lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a}, a + b + c \neq 0$

**Answer :**

$$\begin{aligned}
 \lim_{x \rightarrow 1} \frac{ax^2 + bx + c}{cx^2 + bx + a} &= \frac{a(1)^2 + b(1) + c}{c(1)^2 + b(1) + a} \\
 &= \frac{a + b + c}{a + b + c} \\
 &= 1 \quad [a + b + c \neq 0]
 \end{aligned}$$

**Question 12:**

Evaluate the Given limit:  $\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$

**Answer :**

$$\lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2}$$

At  $x = -2$ , the value of the given function takes the form  $\frac{0}{0}$ .

$$\begin{aligned}
 \text{Now, } \lim_{x \rightarrow -2} \frac{\frac{1}{x} + \frac{1}{2}}{x + 2} &= \lim_{x \rightarrow -2} \frac{\left( \frac{2+x}{2x} \right)}{x + 2} \\
 &= \lim_{x \rightarrow -2} \frac{1}{2x} \\
 &= \frac{1}{2(-2)} = -\frac{1}{4}
 \end{aligned}$$

**Question 13:**

Evaluate the Given limit:  $\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$

**Answer :**

$$\lim_{x \rightarrow 0} \frac{\sin ax}{bx}$$

At  $x = 0$ , the value of the given function takes the form  $\frac{0}{0}$ .

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0} \frac{\sin ax}{bx} &= \lim_{x \rightarrow 0} \frac{\sin ax}{ax} \times \frac{ax}{bx} \\ &= \lim_{x \rightarrow 0} \left( \frac{\sin ax}{ax} \right) \times \left( \frac{a}{b} \right) \\ &= \frac{a}{b} \lim_{ax \rightarrow 0} \left( \frac{\sin ax}{ax} \right) \quad [x \rightarrow 0 \Rightarrow ax \rightarrow 0] \\ &= \frac{a}{b} \times 1 \quad \left[ \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\ &= \frac{a}{b} \end{aligned}$$

**Question 14:**

Evaluate the Given limit:  $\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}$ ,  $a, b \neq 0$

**Answer :**

$$\lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx}, \quad a, b \neq 0$$

At  $x = 0$ , the value of the given function takes the form  $\frac{0}{0}$ .

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow 0} \frac{\sin ax}{\sin bx} &= \lim_{x \rightarrow 0} \frac{\left( \frac{\sin ax}{ax} \right) \times ax}{\left( \frac{\sin bx}{bx} \right) \times bx} \\ &= \left( \frac{a}{b} \right) \times \frac{\lim_{ax \rightarrow 0} \left( \frac{\sin ax}{ax} \right)}{\lim_{bx \rightarrow 0} \left( \frac{\sin bx}{bx} \right)} \quad \left[ \begin{array}{l} x \rightarrow 0 \Rightarrow ax \rightarrow 0 \\ \text{and } x \rightarrow 0 \Rightarrow bx \rightarrow 0 \end{array} \right] \\ &= \left( \frac{a}{b} \right) \times \frac{1}{1} \quad \left[ \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\ &= \frac{a}{b} \end{aligned}$$

**Question 15:**

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

Evaluate the Given limit:

**Answer :**

$$\lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)}$$

It is seen that  $x \rightarrow \pi \Rightarrow (\pi - x) \rightarrow 0$

$$\therefore \lim_{x \rightarrow \pi} \frac{\sin(\pi - x)}{\pi(\pi - x)} = \frac{1}{\pi} \lim_{(\pi - x) \rightarrow 0} \frac{\sin(\pi - x)}{(\pi - x)}$$

$$= \frac{1}{\pi} \times 1 \quad \left[ \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right]$$

$$= \frac{1}{\pi}$$

**Question 16:**

$$\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x}$$

Evaluate the given limit:

**Answer :**

$$\lim_{x \rightarrow 0} \frac{\cos x}{\pi - x} = \frac{\cos 0}{\pi - 0} = \frac{1}{\pi}$$

**Question 17:**

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

Evaluate the Given limit:

**Answer :**

$$\lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1}$$

At  $x = 0$ , the value of the given function takes the form  $\frac{0}{0}$ .

Now,

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{\cos 2x - 1}{\cos x - 1} &= \lim_{x \rightarrow 0} \frac{1 - 2\sin^2 x - 1}{1 - 2\sin^2 \frac{x}{2} - 1} \quad \left[ \cos x = 1 - 2\sin^2 \frac{x}{2} \right] \\
 &= \lim_{x \rightarrow 0} \frac{\sin^2 x}{\sin^2 \frac{x}{2}} = \lim_{x \rightarrow 0} \frac{\left( \frac{\sin^2 x}{x^2} \right) \times x^2}{\left( \frac{\sin^2 \frac{x}{2}}{\left( \frac{x}{2} \right)^2} \right) \times \frac{x^2}{4}} \\
 &= 4 \frac{\lim_{x \rightarrow 0} \left( \frac{\sin^2 x}{x^2} \right)}{\lim_{x \rightarrow 0} \left( \frac{\sin^2 \frac{x}{2}}{\left( \frac{x}{2} \right)^2} \right)} \\
 &= 4 \frac{\left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right)^2}{\left( \lim_{\frac{x}{2} \rightarrow 0} \frac{\sin \frac{x}{2}}{\frac{x}{2}} \right)^2} \quad \left[ x \rightarrow 0 \Rightarrow \frac{x}{2} \rightarrow 0 \right] \\
 &= 4 \frac{1^2}{1^2} \quad \left[ \lim_{y \rightarrow 0} \frac{\sin y}{y} = 1 \right] \\
 &= 4
 \end{aligned}$$

**Question 18:**

Evaluate the Given limit:  $\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$

**Answer :**

$$\lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x}$$

At  $x = 0$ , the value of the given function takes the form  $\frac{0}{0}$ .

Now,

$$\begin{aligned}
 \lim_{x \rightarrow 0} \frac{ax + x \cos x}{b \sin x} &= \frac{1}{b} \lim_{x \rightarrow 0} \frac{x(a + \cos x)}{\sin x} \\
 &= \frac{1}{b} \lim_{x \rightarrow 0} \left( \frac{x}{\sin x} \right) \times \lim_{x \rightarrow 0} (a + \cos x) \\
 &= \frac{1}{b} \times \frac{1}{\left( \lim_{x \rightarrow 0} \frac{\sin x}{x} \right)} \times \lim_{x \rightarrow 0} (a + \cos x) \\
 &= \frac{1}{b} \times (a + \cos 0) \quad \left[ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
 &= \frac{a+1}{b}
 \end{aligned}$$

**Question 19:**

Evaluate the Given limit:  $\lim_{x \rightarrow 0} x \sec x$

**Answer :**

$$\lim_{x \rightarrow 0} x \sec x = \lim_{x \rightarrow 0} \frac{x}{\cos x} = \frac{0}{\cos 0} = \frac{0}{1} = 0$$

**Question 20:**

Evaluate the Given limit:  $\lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx}$   $a, b, a + b \neq 0$

**Answer :**

At  $x = 0$ , the value of the given function takes the form  $\frac{0}{0}$ .  
Now,



$$\begin{aligned}
& \lim_{x \rightarrow 0} \frac{\sin ax + bx}{ax + \sin bx} \\
&= \lim_{x \rightarrow 0} \frac{\left(\frac{\sin ax}{ax}\right) ax + bx}{ax + bx \left(\frac{\sin bx}{bx}\right)} \\
&= \frac{\left(\lim_{ax \rightarrow 0} \frac{\sin ax}{ax}\right) \times \lim_{x \rightarrow 0} (ax) + \lim_{x \rightarrow 0} bx}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx \left(\lim_{bx \rightarrow 0} \frac{\sin bx}{bx}\right)} \quad [\text{As } x \rightarrow 0 \Rightarrow ax \rightarrow 0 \text{ and } bx \rightarrow 0] \\
&= \frac{\lim_{x \rightarrow 0} (ax) + \lim_{x \rightarrow 0} bx}{\lim_{x \rightarrow 0} ax + \lim_{x \rightarrow 0} bx} \quad \left[ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
&= \frac{\lim_{x \rightarrow 0} (ax + bx)}{\lim_{x \rightarrow 0} (ax + bx)} \\
&= \lim_{x \rightarrow 0} (1) \\
&= 1
\end{aligned}$$

**Question 21:**

Evaluate the Given limit:  $\lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x)$

**Answer :**

At  $x = 0$ , the value of the given function takes the form  $\infty - \infty$ .

Now,

$$\begin{aligned}
& \lim_{x \rightarrow 0} (\operatorname{cosec} x - \cot x) \\
&= \lim_{x \rightarrow 0} \left( \frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) \\
&= \lim_{x \rightarrow 0} \left( \frac{1 - \cos x}{\sin x} \right) \\
&= \lim_{x \rightarrow 0} \frac{\left( \frac{1 - \cos x}{x} \right)}{\left( \frac{\sin x}{x} \right)} \\
&= \frac{\lim_{x \rightarrow 0} \frac{1 - \cos x}{x}}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} \\
&= \frac{0}{1} \quad \left[ \lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0 \text{ and } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\
&= 0
\end{aligned}$$

**Question 22:**

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

**Answer :**

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}}$$

At  $x = \frac{\pi}{2}$ , the value of the given function takes the form  $\frac{0}{0}$ .

Now, put  $x - \frac{\pi}{2} = y$  so that  $x \rightarrow \frac{\pi}{2}, y \rightarrow 0$ .

$$\begin{aligned} \therefore \lim_{x \rightarrow \frac{\pi}{2}} \frac{\tan 2x}{x - \frac{\pi}{2}} &= \lim_{y \rightarrow 0} \frac{\tan 2\left(y + \frac{\pi}{2}\right)}{y} \\ &= \lim_{y \rightarrow 0} \frac{\tan(\pi + 2y)}{y} \\ &= \lim_{y \rightarrow 0} \frac{\tan 2y}{y} \quad [\tan(\pi + 2y) = \tan 2y] \\ &= \lim_{y \rightarrow 0} \frac{\sin 2y}{y \cos 2y} \\ &= \lim_{y \rightarrow 0} \left( \frac{\sin 2y}{2y} \times \frac{2}{\cos 2y} \right) \\ &= \left( \lim_{2y \rightarrow 0} \frac{\sin 2y}{2y} \right) \times \lim_{y \rightarrow 0} \left( \frac{2}{\cos 2y} \right) \quad [y \rightarrow 0 \Rightarrow 2y \rightarrow 0] \\ &= 1 \times \frac{2}{\cos 0} \quad \left[ \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \right] \\ &= 1 \times \frac{2}{1} \\ &= 2 \end{aligned}$$

**Question 23:**

Find  $\lim_{x \rightarrow 0} f(x)$  and  $\lim_{x \rightarrow 1} f(x)$ , where  $f(x) = \begin{cases} 2x+3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$

**Answer :**

The given function is

$$f(x) = \begin{cases} 2x+3, & x \leq 0 \\ 3(x+1), & x > 0 \end{cases}$$

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0} [2x + 3] = 2(0) + 3 = 3$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} 3(x + 1) = 3(0 + 1) = 3$$

$$\therefore \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0} f(x) = 3$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} 3(x + 1) = 3(1 + 1) = 6$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} 3(x + 1) = 3(1 + 1) = 6$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = 6$$

**Question 24:**

$$\text{Find } \lim_{x \rightarrow 1} f(x), \text{ where } f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$$

**Answer :**

The given function is

$$f(x) = \begin{cases} x^2 - 1, & x \leq 1 \\ -x^2 - 1, & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} [x^2 - 1] = 1^2 - 1 = 1 - 1 = 0$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} [-x^2 - 1] = -1^2 - 1 = -1 - 1 = -2$$

It is observed that  $\lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$ .

Hence,  $\lim_{x \rightarrow 1} f(x)$  does not exist.

**Question 25:**

$$\text{Evaluate } \lim_{x \rightarrow 0} f(x), \text{ where } f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

**Answer :**

The given function is

$$f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\begin{aligned}\lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \left[ \frac{|x|}{x} \right] \\ &= \lim_{x \rightarrow 0} \left( \frac{-x}{x} \right) \quad \left[ \text{When } x \text{ is negative, } |x| = -x \right] \\ &= \lim_{x \rightarrow 0} (-1) \\ &= -1\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \left[ \frac{|x|}{x} \right] \\ &= \lim_{x \rightarrow 0} \left[ \frac{x}{x} \right] \quad \left[ \text{When } x \text{ is positive, } |x| = x \right] \\ &= \lim_{x \rightarrow 0} (1) \\ &= 1\end{aligned}$$

It is observed that  $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ .

Hence,  $\lim_{x \rightarrow 0} f(x)$  does not exist.

**Question 26:**

$$\lim_{x \rightarrow 0} f(x), \text{ where } f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

**Answer :**

The given function is

$$f(x) = \begin{cases} \frac{x}{|x|}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \left[ \frac{x}{|x|} \right] \\ &= \lim_{x \rightarrow 0^-} \left[ \frac{x}{-x} \right] && [\text{When } x < 0, |x| = -x] \\ &= \lim_{x \rightarrow 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \left[ \frac{x}{|x|} \right] \\ &= \lim_{x \rightarrow 0^+} \left[ \frac{x}{x} \right] && [\text{When } x > 0, |x| = x] \\ &= \lim_{x \rightarrow 0^+} (1) \\ &= 1 \end{aligned}$$

It is observed that  $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ .

Hence,  $\lim_{x \rightarrow 0} f(x)$  does not exist.

**Question 27:**

Find  $\lim_{x \rightarrow 5} f(x)$ , where  $f(x) = |x| - 5$

**Answer :**

The given function is  $f(x) = |x| - 5$ .

$$\begin{aligned} \lim_{x \rightarrow 5^-} f(x) &= \lim_{x \rightarrow 5^-} [|x| - 5] \\ &= \lim_{x \rightarrow 5^-} (x - 5) && [\text{When } x > 0, |x| = x] \\ &= 5 - 5 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 5^+} f(x) &= \lim_{x \rightarrow 5^+} (|x| - 5) \\ &= \lim_{x \rightarrow 5^+} (x - 5) && [\text{When } x > 0, |x| = x] \\ &= 5 - 5 \\ &= 0 \end{aligned}$$

$$\therefore \lim_{x \rightarrow 5^-} f(x) = \lim_{x \rightarrow 5^+} f(x) = 0$$

Hence,  $\lim_{x \rightarrow 5} f(x) = 0$

**Question 28:**

Suppose  $f(x) = \begin{cases} a+bx, & x < 1 \\ 4, & x = 1 \\ b-ax & x > 1 \end{cases}$  and if  $\lim_{x \rightarrow 1} f(x) = f(1)$  what are possible values of  $a$  and  $b$ ?

**Answer :**

The given function is

$$f(x) = \begin{cases} a+bx, & x < 1 \\ 4, & x = 1 \\ b-ax & x > 1 \end{cases}$$

$$\lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1} (a+bx) = a+b$$

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} (b-ax) = b-a$$

$$f(1) = 4$$

It is given that  $\lim_{x \rightarrow 1} f(x) = f(1)$ .

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x) = f(1)$$

$$\Rightarrow a+b = 4 \text{ and } b-a = 4$$

On solving these two equations, we obtain  $a = 0$  and  $b = 4$ .

Thus, the respective possible values of  $a$  and  $b$  are 0 and 4.

**Question 29:**

Let  $a_1, a_2, \dots, a_n$  be fixed real numbers and define a function

$$f(x) = (x-a_1)(x-a_2)\dots(x-a_n)$$

What is  $\lim_{x \rightarrow a_1} f(x)$ ? For some  $a \neq a_1, a_2, \dots, a_n$ , compute  $\lim_{x \rightarrow a} f(x)$ .

**Answer :**

The given function is  $f(x) = (x-a_1)(x-a_2)\dots(x-a_n)$

$$\begin{aligned} \lim_{x \rightarrow a_1} f(x) &= \lim_{x \rightarrow a_1} [(x-a_1)(x-a_2)\dots(x-a_n)] \\ &= \left[ \lim_{x \rightarrow a_1} (x-a_1) \right] \left[ \lim_{x \rightarrow a_1} (x-a_2) \right] \dots \left[ \lim_{x \rightarrow a_1} (x-a_n) \right] \\ &= (a_1-a_1)(a_1-a_2)\dots(a_1-a_n) = 0 \end{aligned}$$

$$\therefore \lim_{x \rightarrow a_1} f(x) = 0$$

$$\begin{aligned} \text{Now, } \lim_{x \rightarrow a} f(x) &= \lim_{x \rightarrow a} [(x-a_1)(x-a_2)\dots(x-a_n)] \\ &= \left[ \lim_{x \rightarrow a} (x-a_1) \right] \left[ \lim_{x \rightarrow a} (x-a_2) \right] \dots \left[ \lim_{x \rightarrow a} (x-a_n) \right] \\ &= (a-a_1)(a-a_2)\dots(a-a_n) \end{aligned}$$

$$\therefore \lim_{x \rightarrow a} f(x) = (a-a_1)(a-a_2)\dots(a-a_n)$$

**Question 30:**

$$\text{If } f(x) = \begin{cases} |x|+1, & x < 0 \\ 0, & x = 0 \\ |x|-1, & x > 0 \end{cases}$$

For what value (s) of  $a$  does  $\lim_{x \rightarrow a} f(x)$  exists?

**Answer :**

The given function is

$$f(x) = \begin{cases} |x|+1, & x < 0 \\ 0, & x = 0 \\ |x|-1, & x > 0 \end{cases}$$

When  $a = 0$ ,

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} (|x|+1) \\ &= \lim_{x \rightarrow 0^-} (-x+1) && [\text{If } x < 0, |x| = -x] \\ &= -0+1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} (|x|-1) \\ &= \lim_{x \rightarrow 0^+} (x-1) && [\text{If } x > 0, |x| = x] \\ &= 0-1 \\ &= -1 \end{aligned}$$

Here, it is observed that  $\lim_{x \rightarrow 0^-} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$ .

$\therefore \lim_{x \rightarrow 0} f(x)$  does not exist.

When  $a < 0$ ,

$$\begin{aligned} \lim_{x \rightarrow a^-} f(x) &= \lim_{x \rightarrow a^-} (|x|+1) \\ &= \lim_{x \rightarrow a^-} (-x+1) && [x < a < 0 \Rightarrow |x| = -x] \\ &= -a+1 \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow a^+} f(x) &= \lim_{x \rightarrow a^+} (|x|+1) \\ &= \lim_{x \rightarrow a^+} (-x+1) && [a < x < 0 \Rightarrow |x| = -x] \\ &= -a+1 \end{aligned}$$

$\therefore \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = -a+1$

Thus, limit of  $f(x)$  exists at  $x = a$ , where  $a < 0$ .

When  $a > 0$

$$\begin{aligned}\lim_{x \rightarrow a^-} f(x) &= \lim_{x \rightarrow a^-} (|x| - 1) \\ &= \lim_{x \rightarrow a^-} (x - 1) \quad [0 < x < a \Rightarrow |x| = x] \\ &= a - 1\end{aligned}$$

$$\begin{aligned}\lim_{x \rightarrow a^+} f(x) &= \lim_{x \rightarrow a^+} (|x| - 1) \\ &= \lim_{x \rightarrow a^+} (x - 1) \quad [0 < a < x \Rightarrow |x| = x] \\ &= a - 1\end{aligned}$$

$$\therefore \lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^+} f(x) = a - 1$$

Thus, limit of  $f(x)$  exists at  $x = a$ , where  $a > 0$ .

Thus,  $\lim_{x \rightarrow a} f(x)$  exists for all  $a \neq 0$ .

**Question 31:**

If the function  $f(x)$  satisfies  $\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} = \pi$ , evaluate  $\lim_{x \rightarrow 1} f(x)$ .

**Answer :**

$$\begin{aligned}\lim_{x \rightarrow 1} \frac{f(x) - 2}{x^2 - 1} &= \pi \\ \Rightarrow \frac{\lim_{x \rightarrow 1} (f(x) - 2)}{\lim_{x \rightarrow 1} (x^2 - 1)} &= \pi \\ \Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) &= \pi \lim_{x \rightarrow 1} (x^2 - 1) \\ \Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) &= \pi(1^2 - 1) \\ \Rightarrow \lim_{x \rightarrow 1} (f(x) - 2) &= 0 \\ \Rightarrow \lim_{x \rightarrow 1} f(x) - \lim_{x \rightarrow 1} 2 &= 0 \\ \Rightarrow \lim_{x \rightarrow 1} f(x) - 2 &= 0 \\ \therefore \lim_{x \rightarrow 1} f(x) &= 2\end{aligned}$$

**Question 32:**

If  $f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$ . For what integers  $m$  and  $n$  does  $\lim_{x \rightarrow 0} f(x)$  and  $\lim_{x \rightarrow 1} f(x)$  exist?

**Answer :**

The given function is



$$f(x) = \begin{cases} mx^2 + n, & x < 0 \\ nx + m, & 0 \leq x \leq 1 \\ nx^3 + m, & x > 1 \end{cases}$$

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0} (mx^2 + n) \\ &= m(0)^2 + n \\ &= n \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0} (nx + m) \\ &= n(0) + m \\ &= m. \end{aligned}$$

Thus,  $\lim_{x \rightarrow 0} f(x)$  exists if  $m = n$ .

$$\begin{aligned} \lim_{x \rightarrow 1^-} f(x) &= \lim_{x \rightarrow 1} (nx + m) \\ &= n(1) + m \\ &= m + n \end{aligned}$$

$$\begin{aligned} \lim_{x \rightarrow 1^+} f(x) &= \lim_{x \rightarrow 1} (nx^3 + m) \\ &= n(1)^3 + m \\ &= m + n \end{aligned}$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1} f(x).$$

Thus,  $\lim_{x \rightarrow 1} f(x)$  exists for any integral value of  $m$  and  $n$ .

## EXCERISE 13.2 SOULTIONS

### Question 1:

Find the derivative of  $x^2 - 2$  at  $x = 10$ .

### Answer :

Let  $f(x) = x^2 - 2$ . Accordingly,

$$\begin{aligned} f'(10) &= \lim_{h \rightarrow 0} \frac{f(10+h) - f(10)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[(10+h)^2 - 2] - (10^2 - 2)}{h} \\ &= \lim_{h \rightarrow 0} \frac{10^2 + 2 \cdot 10 \cdot h + h^2 - 2 - 10^2 + 2}{h} \\ &= \lim_{h \rightarrow 0} \frac{20h + h^2}{h} \\ &= \lim_{h \rightarrow 0} (20 + h) = (20 + 0) = 20 \end{aligned}$$

Thus, the derivative of  $x^2 - 2$  at  $x = 10$  is 20.

### Question 2:

Find the derivative of  $99x$  at  $x = 100$ .

### Answer :

Let  $f(x) = 99x$ . Accordingly,

$$\begin{aligned} f'(100) &= \lim_{h \rightarrow 0} \frac{f(100+h) - f(100)}{h} \\ &= \lim_{h \rightarrow 0} \frac{99(100+h) - 99(100)}{h} \\ &= \lim_{h \rightarrow 0} \frac{99 \times 100 + 99h - 99 \times 100}{h} \\ &= \lim_{h \rightarrow 0} \frac{99h}{h} \\ &= \lim_{h \rightarrow 0} (99) = 99 \end{aligned}$$

Thus, the derivative of  $99x$  at  $x = 100$  is 99.

### Question 3:

Find the derivative of  $x$  at  $x = 1$ .

### Answer :

Let  $f(x) = x$ . Accordingly,

$$\begin{aligned}
 f'(1) &= \lim_{h \rightarrow 0} \frac{f(1+h) - f(1)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(1+h) - 1}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h}{h} \\
 &= \lim_{h \rightarrow 0} (1) \\
 &= 1
 \end{aligned}$$

Thus, the derivative of  $x$  at  $x = 1$  is 1.

**Question 4:**

Find the derivative of the following functions from first principle.

(i)  $x^3 - 27$  (ii)  $(x-1)(x-2)$

(iii)  $\frac{1}{x^2}$  (iv)  $\frac{x+1}{x-1}$

**Answer :**

(i) Let  $f(x) = x^3 - 27$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{[(x+h)^3 - 27] - (x^3 - 27)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{x^3 + h^3 + 3x^2h + 3xh^2 - x^3}{h} \\
 &= \lim_{h \rightarrow 0} \frac{h^3 + 3x^2h + 3xh^2}{h} \\
 &= \lim_{h \rightarrow 0} (h^2 + 3x^2 + 3xh) \\
 &= 0 + 3x^2 + 0 = 3x^2
 \end{aligned}$$

(ii) Let  $f(x) = (x-1)(x-2)$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x+h-1)(x+h-2) - (x-1)(x-2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(x^2 + hx - 2x + hx + h^2 - 2h - x - h + 2) - (x^2 - 2x - x + 2)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{(hx + hx + h^2 - 2h - h)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{2hx + h^2 - 3h}{h} \\
 &= \lim_{h \rightarrow 0} (2x + h - 3) \\
 &= (2x + 0 - 3) \\
 &= 2x - 3
 \end{aligned}$$

(iii) Let  $f(x) = \frac{1}{x^2}$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\frac{1}{(x+h)^2} - \frac{1}{x^2}}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{x^2 - (x+h)^2}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{x^2 - x^2 - h^2 - 2hx}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-h^2 - 2hx}{x^2(x+h)^2} \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{-h - 2x}{x^2(x+h)^2} \right] \\
 &= \frac{0 - 2x}{x^2(x+0)^2} = \frac{-2}{x^3}
 \end{aligned}$$

(iv) Let  $f(x) = \frac{x+1}{x-1}$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\left( \frac{x+h+1}{x+h-1} - \frac{x+1}{x-1} \right)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{(x^2 + hx + x - x - h - 1) - (x^2 + hx - x + x + h - 1)}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2h}{(x-1)(x+h-1)} \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{-2}{(x-1)(x+h-1)} \right] \\
 &= \frac{-2}{(x-1)(x-1)} = \frac{-2}{(x-1)^2}
 \end{aligned}$$

**Question 5:**

For the function

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

Prove that  $f'(1) = 100f'(0)$

**Answer :**

The given function is

$$f(x) = \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left[ \frac{x^{100}}{100} + \frac{x^{99}}{99} + \dots + \frac{x^2}{2} + x + 1 \right]$$

$$\frac{d}{dx} f(x) = \frac{d}{dx} \left( \frac{x^{100}}{100} \right) + \frac{d}{dx} \left( \frac{x^{99}}{99} \right) + \dots + \frac{d}{dx} \left( \frac{x^2}{2} \right) + \frac{d}{dx} (x) + \frac{d}{dx} (1)$$

On using theorem  $\frac{d}{dx}(x^n) = nx^{n-1}$ , we obtain

$$\begin{aligned} \frac{d}{dx} f(x) &= \frac{100x^{99}}{100} + \frac{99x^{98}}{99} + \dots + \frac{2x}{2} + 1 + 0 \\ &= x^{99} + x^{98} + \dots + x + 1 \end{aligned}$$

$$\therefore f'(x) = x^{99} + x^{98} + \dots + x + 1$$

At  $x = 0$ ,

$$f'(0) = 1$$

At  $x = 1$ ,

$$f'(1) = 1^{99} + 1^{98} + \dots + 1 + 1 = [1 + 1 + \dots + 1 + 1]_{100 \text{ terms}} = 1 \times 100 = 100$$

$$\text{Thus, } f'(1) = 100 \times f'(0)$$

**Question 6:**

Find the derivative of  $x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$  for some fixed real number  $a$ .

**Answer :**

$$\text{Let } f(x) = x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n$$

$$\begin{aligned} \therefore f'(x) &= \frac{d}{dx} (x^n + ax^{n-1} + a^2x^{n-2} + \dots + a^{n-1}x + a^n) \\ &= \frac{d}{dx} (x^n) + a \frac{d}{dx} (x^{n-1}) + a^2 \frac{d}{dx} (x^{n-2}) + \dots + a^{n-1} \frac{d}{dx} (x) + a^n \frac{d}{dx} (1) \end{aligned}$$

On using theorem  $\frac{d}{dx} x^n = nx^{n-1}$ , we obtain

$$\begin{aligned} f'(x) &= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} + a^n (0) \\ &= nx^{n-1} + a(n-1)x^{n-2} + a^2(n-2)x^{n-3} + \dots + a^{n-1} \end{aligned}$$

**Question 7:**

For some constants  $a$  and  $b$ , find the derivative of

$$(i) (x-a)(x-b) \quad (ii) (ax^2 + b)^2 \quad (iii) \frac{x-a}{x-b}$$

**Answer :**

$$(i) \text{ Let } f(x) = (x-a)(x-b)$$

$$\Rightarrow f(x) = x^2 - (a+b)x + ab$$

$$\begin{aligned} \therefore f'(x) &= \frac{d}{dx}(x^2 - (a+b)x + ab) \\ &= \frac{d}{dx}(x^2) - (a+b)\frac{d}{dx}(x) + \frac{d}{dx}(ab) \end{aligned}$$

On using theorem  $\frac{d}{dx}(x^n) = nx^{n-1}$ , we obtain

$$f'(x) = 2x - (a+b) + 0 = 2x - a - b$$

(ii) Let  $f(x) = (ax^2 + b)^2$

$$\Rightarrow f(x) = a^2x^4 + 2abx^2 + b^2$$

$$\therefore f'(x) = \frac{d}{dx}(a^2x^4 + 2abx^2 + b^2) = a^2 \frac{d}{dx}(x^4) + 2ab \frac{d}{dx}(x^2) + \frac{d}{dx}(b^2)$$

On using theorem  $\frac{d}{dx}x^n = nx^{n-1}$ , we obtain

$$\begin{aligned} f'(x) &= a^2(4x^3) + 2ab(2x) + b^2(0) \\ &= 4a^2x^3 + 4abx \\ &= 4ax(ax^2 + b) \end{aligned}$$

(iii) Let  $f(x) = \frac{(x-a)}{(x-b)}$

$$\Rightarrow f'(x) = \frac{d}{dx}\left(\frac{x-a}{x-b}\right)$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x-b)\frac{d}{dx}(x-a) - (x-a)\frac{d}{dx}(x-b)}{(x-b)^2} \\ &= \frac{(x-b)(1) - (x-a)(1)}{(x-b)^2} \\ &= \frac{x-b-x+a}{(x-b)^2} \\ &= \frac{a-b}{(x-b)^2} \end{aligned}$$

**Question 8:**

$$\frac{x^n - a^n}{x - a}$$

Find the derivative of  $\frac{x^n - a^n}{x - a}$  for some constant a.

**Answer :**

$$\text{Let } f(x) = \frac{x^n - a^n}{x - a}$$

$$\Rightarrow f'(x) = \frac{d}{dx} \left( \frac{x^n - a^n}{x - a} \right)$$

By quotient rule,

$$\begin{aligned} f'(x) &= \frac{(x-a) \frac{d}{dx}(x^n - a^n) - (x^n - a^n) \frac{d}{dx}(x-a)}{(x-a)^2} \\ &= \frac{(x-a)(nx^{n-1} - 0) - (x^n - a^n)}{(x-a)^2} \\ &= \frac{nx^n - anx^{n-1} - x^n + a^n}{(x-a)^2} \end{aligned}$$

**Question 9:**

Find the derivative of

(i)  $2x - \frac{3}{4}$  (ii)  $(5x^3 + 3x - 1)(x - 1)$

(iii)  $x^{-3}(5 + 3x)$  (iv)  $x^5(3 - 6x^{-9})$

(v)  $x^{-4}(3 - 4x^{-5})$  (vi)  $\frac{2}{x+1} - \frac{x^2}{3x-1}$

**Answer :**

(i) Let  $f(x) = 2x - \frac{3}{4}$

$$\begin{aligned} f'(x) &= \frac{d}{dx} \left( 2x - \frac{3}{4} \right) \\ &= 2 \frac{d}{dx}(x) - \frac{d}{dx} \left( \frac{3}{4} \right) \\ &= 2 - 0 \\ &= 2 \end{aligned}$$

(ii) Let  $f(x) = (5x^3 + 3x - 1)(x - 1)$

By Leibnitz product rule,

$$\begin{aligned} f'(x) &= (5x^3 + 3x - 1) \frac{d}{dx}(x - 1) + (x - 1) \frac{d}{dx}(5x^3 + 3x - 1) \\ &= (5x^3 + 3x - 1)(1) + (x - 1)(15x^2 + 3 - 0) \\ &= (5x^3 + 3x - 1) + (x - 1)(15x^2 + 3) \\ &= 5x^3 + 3x - 1 + 15x^3 + 3x - 15x^2 - 3 \\ &= 20x^3 - 15x^2 + 6x - 4 \end{aligned}$$

(iii) Let  $f(x) = x^{-3}(5 + 3x)$

By Leibnitz product rule,



$$\begin{aligned}
 f'(x) &= x^{-3} \frac{d}{dx}(5+3x) + (5+3x) \frac{d}{dx}(x^{-3}) \\
 &= x^{-3}(0+3) + (5+3x)(-3x^{-3-1}) \\
 &= x^{-3}(3) + (5+3x)(-3x^{-4}) \\
 &= 3x^{-3} - 15x^{-4} - 9x^{-3} \\
 &= -6x^{-3} - 15x^{-4} \\
 &= -3x^{-3} \left( 2 + \frac{5}{x} \right) \\
 &= \frac{-3x^{-3}}{x} (2x+5) \\
 &= \frac{-3}{x^4} (5+2x)
 \end{aligned}$$

(iv) Let  $f(x) = x^5 (3 - 6x^{-9})$

By Leibnitz product rule,

$$\begin{aligned}
 f'(x) &= x^5 \frac{d}{dx}(3-6x^{-9}) + (3-6x^{-9}) \frac{d}{dx}(x^5) \\
 &= x^5 \{0 - 6(-9)x^{-9-1}\} + (3-6x^{-9})(5x^4) \\
 &= x^5 (54x^{-10}) + 15x^4 - 30x^{-5} \\
 &= 54x^{-5} + 15x^4 - 30x^{-5} \\
 &= 24x^{-5} + 15x^4 \\
 &= 15x^4 + \frac{24}{x^5}
 \end{aligned}$$

(v) Let  $f(x) = x^{-4} (3 - 4x^{-5})$

By Leibnitz product rule,

$$\begin{aligned}
 f'(x) &= x^{-4} \frac{d}{dx}(3-4x^{-5}) + (3-4x^{-5}) \frac{d}{dx}(x^{-4}) \\
 &= x^{-4} \{0 - 4(-5)x^{-5-1}\} + (3-4x^{-5})(-4)x^{-4-1} \\
 &= x^{-4} (20x^{-6}) + (3-4x^{-5})(-4x^{-5}) \\
 &= 20x^{-10} - 12x^{-5} + 16x^{-10} \\
 &= 36x^{-10} - 12x^{-5} \\
 &= -\frac{12}{x^5} + \frac{36}{x^{10}} \\
 &= \frac{2}{x^5} - \frac{x^2}{3x-1}
 \end{aligned}$$

(vi) Let  $f(x) = \frac{x+1}{3x-1}$

$$f'(x) = \frac{d}{dx} \left( \frac{2}{x+1} \right) - \frac{d}{dx} \left( \frac{x^2}{3x-1} \right)$$

By quotient rule,

$$\begin{aligned}
 f'(x) &= \left[ \frac{(x+1) \frac{d}{dx}(2) - 2 \frac{d}{dx}(x+1)}{(x+1)^2} \right] - \left[ \frac{(3x-1) \frac{d}{dx}(x^2) - x^2 \frac{d}{dx}(3x-1)}{(3x-1)^2} \right] \\
 &= \left[ \frac{(x+1)(0) - 2(1)}{(x+1)^2} \right] - \left[ \frac{(3x-1)(2x) - (x^2)(3)}{(3x-1)^2} \right] \\
 &= \frac{-2}{(x+1)^2} - \left[ \frac{6x^2 - 2x - 3x^2}{(3x-1)^2} \right] \\
 &= \frac{-2}{(x+1)^2} - \left[ \frac{3x^2 - 2x^2}{(3x-1)^2} \right] \\
 &= \frac{-2}{(x+1)^2} - \frac{x(3x-2)}{(3x-1)^2}
 \end{aligned}$$

**Question 10:**

Find the derivative of  $\cos x$  from first principle.

**Answer :**

Let  $f(x) = \cos x$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\cos(x+h) - \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \left[ \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{-\cos x(1 - \cos h) - \sin x \sin h}{h} \right] \\
 &= \lim_{h \rightarrow 0} \left[ \frac{-\cos x(1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right] \\
 &= -\cos x \left( \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \rightarrow 0} \left( \frac{\sin h}{h} \right) \\
 &= -\cos x(0) - \sin x(1) \quad \left[ \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} = 0 \text{ and } \lim_{h \rightarrow 0} \frac{\sin h}{h} = 1 \right] \\
 &= -\sin x \\
 \therefore f'(x) &= -\sin x
 \end{aligned}$$

**Question 11:**

Find the derivative of the following functions:

- (i)  $\sin x \cos x$  (ii)  $\sec x$  (iii)  $5 \sec x + 4 \cos x$   
 (iv)  $\operatorname{cosec} x$  (v)  $3 \cot x + 5 \operatorname{cosec} x$   
 (vi)  $5 \sin x - 6 \cos x + 7$  (vii)  $2 \tan x - 7 \sec x$

**Answer :**

(i) Let  $f(x) = \sin x \cos x$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sin(x+h)\cos(x+h) - \sin x \cos x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [2\sin(x+h)\cos(x+h) - 2\sin x \cos x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} [\sin 2(x+h) - \sin 2x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{2h} \left[ 2\cos \frac{2x+2h+2x}{2} \cdot \sin \frac{2x+2h-2x}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \cos \frac{4x+2h}{2} \sin \frac{2h}{2} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [\cos(2x+h)\sin h] \\
 &= \lim_{h \rightarrow 0} \cos(2x+h) \cdot \lim_{h \rightarrow 0} \frac{\sin h}{h} \\
 &= \cos(2x+0) \cdot 1 \\
 &= \cos 2x
 \end{aligned}$$

(ii) Let  $f(x) = \sec x$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{\sec(x+h) - \sec x}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{h \rightarrow 0} \left[ \frac{\sin\left(\frac{2x+h}{2}\right) \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\cos(x+h)} \right] \\
 &= \frac{1}{\cos x} \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \\
 &= \frac{1}{\cos x} \cdot 1 \cdot \frac{\sin x}{\cos x} \\
 &= \sec x \tan x
 \end{aligned}$$

(iii) Let  $f(x) = 5 \sec x + 4 \cos x$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{5 \sec(x+h) + 4 \cos(x+h) - [5 \sec x + 4 \cos x]}{h} \\
 &= 5 \lim_{h \rightarrow 0} \frac{[\sec(x+h) - \sec x]}{h} + 4 \lim_{h \rightarrow 0} \frac{[\cos(x+h) - \cos x]}{h} \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [\cos x \cos h - \sin x \sin h - \cos x] \\
 &= \frac{5}{\cos x} \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos(x+h)} \right] + 4 \lim_{h \rightarrow 0} \frac{1}{h} [-\cos x(1 - \cos h) - \sin x \sin h] \\
 &= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos(x+h)} \right] + 4 \left[ -\cos x \lim_{h \rightarrow 0} \frac{(1 - \cos h)}{h} - \sin x \lim_{h \rightarrow 0} \frac{\sin h}{h} \right] \\
 &= \frac{5}{\cos x} \cdot \lim_{h \rightarrow 0} \left[ \frac{\sin\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}}}{\cos(x+h)} \right] + 4 [(-\cos x) \cdot (0) - (\sin x) \cdot 1] \\
 &= \frac{5}{\cos x} \cdot \left[ \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos(x+h)} \cdot \lim_{h \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} \right] - 4 \sin x \\
 &= \frac{5}{\cos x} \cdot \frac{\sin x}{\cos x} \cdot 1 - 4 \sin x \\
 &= 5 \sec x \tan x - 4 \sin x
 \end{aligned}$$

(iv) Let  $f(x) = \operatorname{cosec} x$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 f'(x) &= \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin x - \sin(x+h)}{\sin(x+h) \sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{2 \cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h) \sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\sin(x+h) \sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{-\cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h) \sin x} \\
 &= \lim_{h \rightarrow 0} \left( \frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h) \sin x} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
 &= \left( \frac{-\cos x}{\sin x \sin x} \right) \cdot 1 \\
 &= -\operatorname{cosec} x \cot x
 \end{aligned}$$

(v) Let  $f(x) = 3\cot x + 5\operatorname{cosec} x$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{3 \cot(x+h) + 5 \operatorname{cosec}(x+h) - 3 \cot x - 5 \operatorname{cosec} x}{h} \\
 &= 3 \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] + 5 \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \quad \dots(1)
 \end{aligned}$$

$$\begin{aligned}
 \text{Now, } & \lim_{h \rightarrow 0} \frac{1}{h} [\cot(x+h) - \cot x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\cos(x+h)}{\sin(x+h)} - \frac{\cos x}{\sin x} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\cos(x+h) \sin x - \cos x \sin(x+h)}{\sin x \sin(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x-x-h)}{\sin x \sin(x+h)} \right] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(-h)}{\sin x \sin(x+h)} \right] \\
 &= - \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \cdot \left( \lim_{h \rightarrow 0} \frac{1}{\sin x \cdot \sin(x+h)} \right) \\
 &= -1 \cdot \frac{1}{\sin x \cdot \sin(x+0)} = \frac{-1}{\sin^2 x} = -\operatorname{cosec}^2 x \quad \dots(2)
 \end{aligned}$$

$$\begin{aligned}
& \lim_{h \rightarrow 0} \frac{1}{h} [\operatorname{cosec}(x+h) - \operatorname{cosec} x] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{\sin(x+h)} - \frac{1}{\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin x - \sin(x+h)}{\sin(x+h)\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{2 \cos\left(\frac{x+x+h}{2}\right) \cdot \sin\left(\frac{x-x-h}{2}\right)}{\sin(x+h)\sin x} \right] \\
&= \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{2 \cos\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\sin(x+h)\sin x} \right] \\
&\quad - \cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
&= \lim_{h \rightarrow 0} \frac{-\cos\left(\frac{2x+h}{2}\right) \cdot \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}}{\sin(x+h)\sin x} \\
&= \lim_{h \rightarrow 0} \left( \frac{-\cos\left(\frac{2x+h}{2}\right)}{\sin(x+h)\sin x} \right) \cdot \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin\left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)} \\
&= \left( \frac{-\cos x}{\sin x \sin x} \right) \cdot 1 \\
&= -\operatorname{cosec} x \cot x \quad \dots(3)
\end{aligned}$$

From (1), (2), and (3), we obtain

$$f'(x) = -3\operatorname{cosec}^2 x - 5\operatorname{cosec} x \cot x$$

(vi) Let  $f(x) = 5\sin x - 6\cos x + 7$ . Accordingly, from the first principle,



$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [5 \sin(x+h) - 6 \cos(x+h) + 7 - 5 \sin x + 6 \cos x - 7] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [5 \{ \sin(x+h) - \sin x \} - 6 \{ \cos(x+h) - \cos x \}] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} [\sin(x+h) - \sin x] - 6 \lim_{h \rightarrow 0} \frac{1}{h} [\cos(x+h) - \cos x] \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[ 2 \cos \left( \frac{x+h+x}{2} \right) \sin \left( \frac{x+h-x}{2} \right) \right] - 6 \lim_{h \rightarrow 0} \frac{\cos x \cos h - \sin x \sin h - \cos x}{h} \\
 &= 5 \lim_{h \rightarrow 0} \frac{1}{h} \left[ 2 \cos \left( \frac{2x+h}{2} \right) \sin \frac{h}{2} \right] - 6 \lim_{h \rightarrow 0} \left[ \frac{-\cos x (1 - \cos h) - \sin x \sin h}{h} \right] \\
 &= 5 \lim_{h \rightarrow 0} \left[ \cos \left( \frac{2x+h}{2} \right) \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \lim_{h \rightarrow 0} \left[ \frac{-\cos x (1 - \cos h)}{h} - \frac{\sin x \sin h}{h} \right] \\
 &= 5 \left[ \lim_{h \rightarrow 0} \cos \left( \frac{2x+h}{2} \right) \right] \left[ \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right] - 6 \left[ (-\cos x) \left( \lim_{h \rightarrow 0} \frac{1 - \cos h}{h} \right) - \sin x \lim_{h \rightarrow 0} \left( \frac{\sin h}{h} \right) \right] \\
 &= 5 \cos x \cdot 1 - 6 [(-\cos x) \cdot (0) - \sin x \cdot 1] \\
 &= 5 \cos x + 6 \sin x
 \end{aligned}$$

(vii) Let  $f(x) = 2 \tan x - 7 \sec x$ . Accordingly, from the first principle,

$$\begin{aligned}
 f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [2 \tan(x+h) - 7 \sec(x+h) - 2 \tan x + 7 \sec x] \\
 &= \lim_{h \rightarrow 0} \frac{1}{h} [2 \{ \tan(x+h) - \tan x \} - 7 \{ \sec(x+h) - \sec x \}] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} [\tan(x+h) - \tan x] - 7 \lim_{h \rightarrow 0} \frac{1}{h} [\sec(x+h) - \sec x] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h)}{\cos(x+h)} - \frac{\sin x}{\cos x} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{1}{\cos(x+h)} - \frac{1}{\cos x} \right] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h) \cos x - \sin x \cos(x+h)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\cos x - \cos(x+h)}{\cos x \cos(x+h)} \right] \\
 &= 2 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{\sin(x+h-x)}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2 \sin\left(\frac{x+x+h}{2}\right) \sin\left(\frac{x-x-h}{2}\right)}{\cos x \cos(x+h)} \right] \\
 &= 2 \lim_{h \rightarrow 0} \left[ \left( \frac{\sin h}{h} \right) \frac{1}{\cos x \cos(x+h)} \right] - 7 \lim_{h \rightarrow 0} \frac{1}{h} \left[ \frac{-2 \sin\left(\frac{2x+h}{2}\right) \sin\left(-\frac{h}{2}\right)}{\cos x \cos(x+h)} \right] \\
 &= 2 \left( \lim_{h \rightarrow 0} \frac{\sin h}{h} \right) \left( \lim_{h \rightarrow 0} \frac{1}{\cos x \cos(x+h)} \right) - 7 \left( \lim_{\frac{h}{2} \rightarrow 0} \frac{\sin \frac{h}{2}}{\frac{h}{2}} \right) \left( \lim_{h \rightarrow 0} \frac{\sin\left(\frac{2x+h}{2}\right)}{\cos x \cos(x+h)} \right) \\
 &= 2 \cdot 1 \cdot \frac{1}{\cos x \cos x} - 7 \cdot 1 \left( \frac{\sin x}{\cos x \cos x} \right) \\
 &= 2 \sec^2 x - 7 \sec x \tan x
 \end{aligned}$$