Exercise 4.1

Question 1:

Evaluate the determinants in Exercises 1 and 2.

$$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$$

Answer

$\begin{vmatrix} 2 & 4 \\ -5 & -1 \end{vmatrix}$ = 2(−1) − 4(−5) = − 2 + 20 = 18

Question 2:

Evaluate the determinants in Exercises 1 and 2.

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ (ii) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

Answer

(i) $\begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$ = $(\cos\theta)(\cos\theta) - (-\sin\theta)(\sin\theta) = \cos^2\theta + \sin^2\theta = 1$

(ii) $\begin{vmatrix} x^2-x+1 & x-1 \\ x+1 & x+1 \end{vmatrix}$

$= (x^2 - x + 1)(x + 1) - (x - 1)(x + 1)$

$= x^3 - x^2 + x + x^2 - x + 1 - (x^2 - 1)$

$= x^3 + 1 - x^2 + 1$

$= x^3 - x^2 + 2$

Question 3:

If $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$, then show that $|2A| = 4|A|$

Answer

The given matrix is $A = \begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix}$.

$$\therefore 2A = 2\begin{bmatrix} 1 & 2 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 8 & 4 \end{bmatrix}$$

$$\therefore \text{L.H.S.} = |2A| = \begin{vmatrix} 2 & 4 \\ 8 & 4 \end{vmatrix} = 2\times 4 - 4\times 8 = 8 - 32 = -24$$

$$\text{Now, } |A| = \begin{vmatrix} 1 & 2 \\ 4 & 2 \end{vmatrix} = 1\times 2 - 2\times 4 = 2 - 8 = -6$$

$$\therefore \text{R.H.S.} = 4|A| = 4\times(-6) = -24$$

$$\therefore \text{L.H.S.} = \text{R.H.S.}$$

Question 4:

If $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, then show that $|3A| = 27|A|$.

Answer

The given matrix is $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$.

It can be observed that in the first column, two entries are zero. Thus, we expand along the first column (C_1) for easier calculation.

$$|A| = 1\begin{vmatrix} 1 & 2 \\ 0 & 4 \end{vmatrix} - 0\begin{vmatrix} 0 & 1 \\ 0 & 4 \end{vmatrix} + 0\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} = 1(4-0) - 0 + 0 = 4$$

$$\therefore 27|A| = 27(4) = 108 \qquad \text{...(i)}$$

$$\text{Now, } 3A = 3\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 12 \end{bmatrix}$$

$$\therefore |3A| = 3\begin{vmatrix} 3 & 6 \\ 0 & 12 \end{vmatrix} - 0\begin{vmatrix} 0 & 3 \\ 0 & 12 \end{vmatrix} + 0\begin{vmatrix} 0 & 3 \\ 3 & 6 \end{vmatrix}$$

$$= 3(36-0) = 3(36) = 108 \qquad \text{...(ii)}$$

From equations (i) and (ii), we have:

$$|3A| = 27|A|$$

Hence, the given result is proved.

Question 5:

Evaluate the determinants

(i) $\begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$ (iii) $\begin{vmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{vmatrix}$

(ii) $\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ -2 & 3 & 0 \end{vmatrix}$ (iv) $\begin{bmatrix} 2 & -1 & -2 \\ 0 & 2 & -1 \\ 3 & -5 & 0 \end{bmatrix}$

Answer

(i) Let $A = \begin{vmatrix} 3 & -1 & -2 \\ 0 & 0 & -1 \\ 3 & -5 & 0 \end{vmatrix}$.

It can be observed that in the second row, two entries are zero. Thus, we expand along the second row for easier calculation.

$$|A| = -0\begin{vmatrix} -1 & -2 \\ -5 & 0 \end{vmatrix} + 0\begin{vmatrix} 3 & -2 \\ 3 & 0 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -1 \\ 3 & -5 \end{vmatrix} = (-15+3) = -12$$

(ii) Let $A = \begin{bmatrix} 3 & -4 & 5 \\ 1 & 1 & -2 \\ 2 & 3 & 1 \end{bmatrix}$.

By expanding along the first row, we have:

$$|A| = 3\begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix} + 4\begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix} + 5\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 3(1+6) + 4(1+4) + 5(3-2)$$

$$= 3(7) + 4(5) + 5(1)$$

$$= 21 + 20 + 5 = 46$$

(iii) Let $A=\begin{bmatrix}0 & 1 & 2\\ -1 & 0 & -3\\ -2 & 3 & 0\end{bmatrix}$.

By expanding along the first row, we have:

$$|A|=0\begin{vmatrix}0 & -3\\ 3 & 0\end{vmatrix}-1\begin{vmatrix}-1 & -3\\ -2 & 0\end{vmatrix}+2\begin{vmatrix}-1 & 0\\ -2 & 3\end{vmatrix}$$
$$=0-1(0-6)+2(-3-0)$$
$$=-1(-6)+2(-3)$$
$$=6-6=0$$

(iv) Let $A=\begin{bmatrix}2 & -1 & -2\\ 0 & 2 & -1\\ 3 & -5 & 0\end{bmatrix}$.

By expanding along the first column, we have:

$$|A|=2\begin{vmatrix}2 & -1\\ -5 & 0\end{vmatrix}-0\begin{vmatrix}-1 & -2\\ -5 & 0\end{vmatrix}+3\begin{vmatrix}-1 & -2\\ 2 & -1\end{vmatrix}$$
$$=2(0-5)-0+3(1+4)$$
$$=-10+15=5$$

Question 6:

If $A=\begin{bmatrix}1 & 1 & -2\\ 2 & 1 & -3\\ 5 & 4 & -9\end{bmatrix}$, find $|\mathrm{A}|$.

Answer

Let $A=\begin{bmatrix}1 & 1 & -2\\ 2 & 1 & -3\\ 5 & 4 & -9\end{bmatrix}$.

By expanding along the first row, we have:

$$|A| = 1\begin{vmatrix} 1 & -3 \\ 4 & -9 \end{vmatrix} - 1\begin{vmatrix} 2 & -3 \\ 5 & -9 \end{vmatrix} - 2\begin{vmatrix} 2 & 1 \\ 5 & 4 \end{vmatrix}$$

$$= 1(-9+12) - 1(-18+15) - 2(8-5)$$

$$= 1(3) - 1(-3) - 2(3)$$

$$= 3 + 3 - 6$$

$$= 6 - 6$$

$$= 0$$

Question 7:

Find values of x, if

(i) $\begin{vmatrix} 2 & 4 \\ 2 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$ (ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

Answer

(i) $\begin{vmatrix} 2 & 4 \\ 5 & 1 \end{vmatrix} = \begin{vmatrix} 2x & 4 \\ 6 & x \end{vmatrix}$

$$\Rightarrow 2\times1 - 5\times4 = 2x\times x - 6\times4$$

$$\Rightarrow 2 - 20 = 2x^2 - 24$$

$$\Rightarrow 2x^2 = 6$$

$$\Rightarrow x^2 = 3$$

$$\Rightarrow x = \pm\sqrt{3}$$

(ii) $\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} = \begin{vmatrix} x & 3 \\ 2x & 5 \end{vmatrix}$

$$\Rightarrow 2\times5 - 3\times4 = x\times5 - 3\times2x$$

$$\Rightarrow 10 - 12 = 5x - 6x$$

$$\Rightarrow -2 = -x$$

$$\Rightarrow x = 2$$

Question 8:

If $\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$, then x is equal to

(A) 6 (B) ±6 (C) −6 (D) 0

Answer

Answer: B

$$\begin{vmatrix} x & 2 \\ 18 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 18 & 6 \end{vmatrix}$$

$$\Rightarrow x^2 - 36 = 36 - 36$$

$$\Rightarrow x^2 - 36 = 0$$

$$\Rightarrow x^2 = 36$$

$$\Rightarrow x = \pm 6$$

Hence, the correct answer is B.

Exercise 4.2

Question 1:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = 0$$

Answer

$$\begin{vmatrix} x & a & x+a \\ y & b & y+b \\ z & c & z+c \end{vmatrix} = \begin{vmatrix} x & a & x \\ y & b & y \\ z & c & z \end{vmatrix} + \begin{vmatrix} x & a & a \\ y & b & b \\ z & c & c \end{vmatrix} = 0+0=0$$

[Here, the two columns of the determinants are identical]

Question 2:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$, we have:

$$\Delta = \begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ -(a-c) & -(b-a) & -(c-b) \end{vmatrix}$$

$$= -\begin{vmatrix} a-c & b-a & c-b \\ b-c & c-a & a-b \\ a-c & b-a & c-b \end{vmatrix}$$

Here, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0$.

Question 3:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = 0$$

Answer

$$\begin{vmatrix} 2 & 7 & 65 \\ 3 & 8 & 75 \\ 5 & 9 & 86 \end{vmatrix} = \begin{vmatrix} 2 & 7 & 63+2 \\ 3 & 8 & 72+3 \\ 5 & 9 & 81+5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 63 \\ 3 & 8 & 72 \\ 5 & 9 & 81 \end{vmatrix} + \begin{vmatrix} 2 & 7 & 2 \\ 3 & 8 & 3 \\ 5 & 9 & 5 \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 7 & 9(7) \\ 3 & 8 & 9(8) \\ 5 & 9 & 9(9) \end{vmatrix} + 0 \qquad [\text{Two columns are identical}]$$

$$= 9\begin{vmatrix} 2 & 7 & 7 \\ 3 & 8 & 8 \\ 5 & 9 & 9 \end{vmatrix}$$

$$= 0 \qquad [\text{Two columns are identical}]$$

Question 4:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$$

By applying $C_3 \rightarrow C_3 + C_2$, we have:

$$\Delta = \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$$

Here, two columns C_1 and C_3 are proportional.

$\therefore \Delta = 0$.

Question 5:

Using the property of determinants and without expanding, prove that:

$$\begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix} = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Answer

$$\Delta = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a+b & p+q & x+y \end{vmatrix}$$

$$= \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix} + \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

$$= \Delta_1 + \Delta_2 \text{ (say)} \qquad ...(1)$$

$$\text{Now, } \Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ a & p & x \end{vmatrix}$$

Applying $R_2 \rightarrow R_2 - R_3$, we have:

$$\Delta_1 = \begin{vmatrix} b+c & q+r & y+z \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying $R_1 \rightarrow R_1 - R_2$, we have:

$$\Delta_1 = \begin{vmatrix} b & q & y \\ c & r & z \\ a & p & x \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_3$ and $R_2 \leftrightarrow R_3$, we have:

$$\Delta_1 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad ...(2)$$

$$\Delta_2 = \begin{vmatrix} b+c & q+r & y+z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \to R_1 - R_3$, we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ c+a & r+p & z+x \\ b & q & y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$, we have:

$$\Delta_2 = \begin{vmatrix} c & r & z \\ a & p & x \\ b & q & y \end{vmatrix}$$

Applying $R_1 \leftrightarrow R_2$ and $R_2 \leftrightarrow R_3$, we have:

$$\Delta_2 = (-1)^2 \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = \begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} \qquad ...(3)$$

From (1), (2), and (3), we have:

$$\Delta = 2\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix}$$

Hence, the given result is proved.

Question 6:

By using properties of determinants, show that:

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & -c \\ b & c & 0 \end{vmatrix} = 0$$

Answer

We have,

$$\Delta=\begin{vmatrix}0 & a & -b\\ -a & 0 & -c\\ b & c & 0\end{vmatrix}$$

Applying $R_1 \to cR_1$, we have:

$$\Delta=\frac{1}{c}\begin{vmatrix}0 & ac & -bc\\ -a & 0 & -c\\ b & c & 0\end{vmatrix}$$

Applying $R_1 \to R_1 - bR_2$, we have:

$$\Delta=\frac{1}{c}\begin{vmatrix}ab & ac & 0\\ -a & 0 & -c\\ b & c & 0\end{vmatrix}$$

$$=\frac{a}{c}\begin{vmatrix}b & c & 0\\ -a & 0 & -c\\ b & c & 0\end{vmatrix}$$

Here, the two rows R_1 and R_3 are identical.

$\therefore \Delta = 0$.

Question 7:

By using properties of determinants, show that:

$$\begin{vmatrix}-a^2 & ab & ac\\ ba & -b^2 & bc\\ ca & cb & -c^2\end{vmatrix}=4a^2b^2c^2$$

Answer

$$\Delta = \begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ca & cb & -c^2 \end{vmatrix}$$

$$= abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \qquad \left[\text{Taking out factors } a,\ b,\ c \text{ from } R_1,\ R_2, \text{ and } R_3\right]$$

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \qquad \left[\text{Taking out factors } a,\ b,\ c \text{ from } C_1,\ C_2, \text{ and } C_3\right]$$

Applying $R_2 \to R_2 + R_1$ and $R_3 \to R_3 + R_1$, we have:

$$\Delta = a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 2 & 0 \end{vmatrix}$$

$$= a^2b^2c^2(-1)\begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -a^2b^2c^2(0-4) = 4a^2b^2c^2$$

Question 8:

By using properties of determinants, show that:

(i) $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

(ii) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$

Answer

(i) $\text{Let } \Delta = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}.$

Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, we have:

$$\Delta = \begin{vmatrix} 0 & a-c & a^2-c^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (c-a)(b-c)\begin{vmatrix} 0 & -1 & -a-c \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2$, we have:

$$\Delta = (b-c)(c-a)\begin{vmatrix} 0 & 0 & -a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\begin{vmatrix} 0 & 0 & -1 \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = (a-b)(b-c)(c-a)\begin{vmatrix} 0 & -1 \\ 1 & b+c \end{vmatrix} = (a-b)(b-c)(c-a)$$

Hence, the given result is proved.

(ii) Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}$.

Applying $C_1 \to C_1 - C_3$ and $C_2 \to C_2 - C_3$, we have:

$$\Delta=\begin{vmatrix}0 & 0 & 1\\ a-c & b-c & c\\ a^3-c^3 & b^3-c^3 & c^3\end{vmatrix}$$

$$=\begin{vmatrix}0 & 0 & 1\\ a-c & b-c & c\\ (a-c)(a^2+ac+c^2) & (b-c)(b^2+bc+c^2) & c^3\end{vmatrix}$$

$$=(c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ -1 & 1 & c\\ -(a^2+ac+c^2) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

Applying $C_1 \to C_1 + C_2$, we have:

$$\Delta=(c-a)(b-c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ (b^2-a^2)+(bc-ac) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$=(b-c)(c-a)(a-b)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ -(a+b+c) & (b^2+bc+c^2) & c^3\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)\begin{vmatrix}0 & 0 & 1\\ 0 & 1 & c\\ -1 & (b^2+bc+c^2) & c^3\end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta=(a-b)(b-c)(c-a)(a+b+c)(-1)\begin{vmatrix}0 & 1\\ 1 & c\end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)$$

Hence, the given result is proved.

Question 9:

By using properties of determinants, show that:

$$\begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix} = (x-y)(y-z)(z-x)(xy+yz+zx)$$

Answer

$$\text{Let } \Delta = \begin{vmatrix} x & x^2 & yz \\ y & y^2 & zx \\ z & z^2 & xy \end{vmatrix}.$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} x & x^2 & yz \\ y-x & y^2-x^2 & zx-yz \\ z-x & z^2-x^2 & xy-yz \end{vmatrix}$$

$$= \begin{vmatrix} x & x^2 & yz \\ -(x-y) & -(x-y)(x+y) & z(x-y) \\ (z-x) & (z-x)(z+x) & -y(z-x) \end{vmatrix}$$

$$= (x-y)(z-x) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 1 & z+x & -y \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta = (x-y)(z-x) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & z-y & z-y \end{vmatrix}$$

$$= (x-y)(z-x)(z-y) \begin{vmatrix} x & x^2 & yz \\ -1 & -x-y & z \\ 0 & 1 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = \left[(x-y)(z-x)(z-y)\right]\left[(-1)\begin{vmatrix} x & yz \\ -1 & z \end{vmatrix} + 1\begin{vmatrix} x & x^2 \\ -1 & -x-y \end{vmatrix}\right]$$

$$= (x-y)(z-x)(z-y)\left[(-xz-yz)+(-x^2-xy+x^2)\right]$$

$$= -(x-y)(z-x)(z-y)(xy+yz+zx)$$

$$= (x-y)(y-z)(z-x)(xy+yz+zx)$$

Hence, the given result is proved.

Question 10:

By using properties of determinants, show that:

(i) $\begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix} = (5x+4)(4-x)^2$

(ii) $\begin{vmatrix} y+k & y & y \\ y & y+k & y \\ y & y & y+k \end{vmatrix} = k^2(3y+k)$

Answer

(i) $\Delta = \begin{vmatrix} x+4 & 2x & 2x \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 5x+4 & 5x+4 & 5x+4 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

$$= (5x+4)\begin{vmatrix} 1 & 1 & 1 \\ 2x & x+4 & 2x \\ 2x & 2x & x+4 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have:

$$\Delta=(5x+4)\begin{vmatrix}1 & 0 & 0\\ 2x & -x+4 & 0\\ 2x & 0 & -x+4\end{vmatrix}$$

$$=(5x+4)(4-x)(4-x)\begin{vmatrix}1 & 0 & 0\\ 2x & 1 & 0\\ 2x & 0 & 1\end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta=(5x+4)(4-x)^2\begin{vmatrix}1 & 0\\ 2x & 1\end{vmatrix}$$

$$=(5x+4)(4-x)^2$$

Hence, the given result is proved.

(ii) $$\Delta=\begin{vmatrix}y+k & y & y\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta=\begin{vmatrix}3y+k & 3y+k & 3y+k\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$$

$$=(3y+k)\begin{vmatrix}1 & 1 & 1\\ y & y+k & y\\ y & y & y+k\end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta=(3y+k)\begin{vmatrix}1 & 0 & 0\\ y & k & 0\\ y & 0 & k\end{vmatrix}$$

$$=k^2(3y+k)\begin{vmatrix}1 & 0 & 0\\ y & 1 & 0\\ y & 0 & 1\end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = k^2(3y+k)\begin{vmatrix} 1 & 0 \\ y & 1 \end{vmatrix} = k^2(3y+k)$$

Hence, the given result is proved.

Question 11:

By using properties of determinants, show that:

(i) $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3$

(ii) $\begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix} = 2(x+y+z)^3$

Answer

(i) $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$, we have:

$$\Delta = (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -(a+b+c) & 0 \\ 2c & 0 & -(a+b+c) \end{vmatrix}$$

$$= (a+b+c)^3\begin{vmatrix} 1 & 0 & 0 \\ 2b & -1 & 0 \\ 2c & 0 & -1 \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = (a+b+c)^3(-1)(-1) = (a+b+c)^3$$

Hence, the given result is proved.

(ii)
$$\Delta = \begin{vmatrix} x+y+2z & x & y \\ z & y+z+2x & y \\ z & x & z+x+2y \end{vmatrix}$$

Applying $C_1 \to C_1 + C_2 + C_3$, we have:

$$\Delta = \begin{vmatrix} 2(x+y+z) & x & y \\ 2(x+y+z) & y+z+2x & y \\ 2(x+y+z) & x & z+x+2y \end{vmatrix}$$

$$= 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 1 & y+z+2x & y \\ 1 & x & z+x+2y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = 2(x+y+z)\begin{vmatrix} 1 & x & y \\ 0 & x+y+z & 0 \\ 0 & 0 & x+y+z \end{vmatrix}$$

$$= 2(x+y+z)^3\begin{vmatrix} 1 & x & y \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = 2(x+y+z)^3(1)(1-0) = 2(x+y+z)^3$$

Hence, the given result is proved.

Question 12:

By using properties of determinants, show that:

$$\begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix} = (1-x^3)^2$$

Answer

$$\Delta = \begin{vmatrix} 1 & x & x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 1+x+x^2 & 1+x+x^2 & 1+x+x^2 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

$$= \left(1+x+x^2\right)\begin{vmatrix} 1 & 1 & 1 \\ x^2 & 1 & x \\ x & x^2 & 1 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = \left(1+x+x^2\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1-x^2 & x-x^2 \\ x & x^2-x & 1-x \end{vmatrix}$$

$$= \left(1+x+x^2\right)\left(1-x\right)\left(1-x\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)\left(1-x\right)\begin{vmatrix} 1 & 0 & 0 \\ x^2 & 1+x & x \\ x & -x & 1 \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = \left(1-x^3\right)\left(1-x\right)\left(1\right)\begin{vmatrix} 1+x & x \\ -x & 1 \end{vmatrix}$$

$$= \left(1-x^3\right)\left(1-x\right)\left(1+x+x^2\right)$$

$$= \left(1-x^3\right)\left(1-x^3\right)$$

$$= \left(1-x^3\right)^2$$

Hence, the given result is proved.

Question 13:

By using properties of determinants, show that:

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = \left(1+a^2+b^2\right)^3$$

Answer

$$\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Applying $R_1 \to R_1 + bR_3$ and $R_2 \to R_2 - aR_3$, we have:

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -b\left(1+a^2+b^2\right) \\ 0 & 1+a^2+b^2 & a\left(1+a^2+b^2\right) \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 0 & -b \\ 0 & 1 & a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = \left(1+a^2+b^2\right)^2 \left[(1)\begin{vmatrix} 1 & a \\ -2a & 1-a^2-b^2 \end{vmatrix} - b\begin{vmatrix} 0 & 1 \\ 2b & -2a \end{vmatrix}\right]$$

$$= \left(1+a^2+b^2\right)^2 \left[1-a^2-b^2+2a^2-b(-2b)\right]$$

$$= \left(1+a^2+b^2\right)^2 \left(1+a^2+b^2\right)$$

$$= \left(1+a^2+b^2\right)^3$$

Question 14:

By using properties of determinants, show that:

$$\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$$

Answer

$$\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ca & cb & c^2+1 \end{vmatrix}$$

Taking out common factors a, b, and c from R_1, R_2, and R_3 respectively, we have:

$$\Delta = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ a & b+\frac{1}{b} & c \\ a & b & c+\frac{1}{c} \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = abc\begin{vmatrix} a+\frac{1}{a} & b & c \\ -\frac{1}{a} & \frac{1}{b} & 0 \\ -\frac{1}{a} & 0 & \frac{1}{c} \end{vmatrix}$$

Applying $C_1 \to aC_1$, $C_2 \to bC_2$, and $C_3 \to cC_3$, we have:

$$\Delta = abc \times \frac{1}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} a^2+1 & b^2 & c^2 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = -1\begin{vmatrix} b^2 & c^2 \\ 1 & 0 \end{vmatrix} + 1\begin{vmatrix} a^2+1 & b^2 \\ -1 & 1 \end{vmatrix}$$

$$= -1\left(-c^2\right) + \left(a^2+1+b^2\right) = 1+a^2+b^2+c^2$$

Hence, the given result is proved.

Question 15:

Choose the correct answer.

Let A be a square matrix of order 3 × 3, then $|kA|$ is equal to

A. $k|A|$ **B.** $k^2|A|$ **C.** $k^3|A|$ **D.** $3k|A|$

Answer

Answer: C

A is a square matrix of order 3 × 3.

$$\text{Let } A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.$$

$$\text{Then, } kA = \begin{bmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}.$$

$$\therefore |kA| = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ ka_2 & kb_2 & kc_2 \\ ka_3 & kb_3 & kc_3 \end{vmatrix}$$

$$= k^3\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad (\text{Taking out common factors } k \text{ from each row})$$

$$= k^3|A|$$

$$\therefore |kA| = k^3|A|$$

Hence, the correct answer is C.

Question 16:

Which of the following is correct?

A. Determinant is a square matrix.

B. Determinant is a number associated to a matrix.

C. Determinant is a number associated to a square matrix.

D. None of these

Answer

Answer: C

We know that to every square matrix, $A = [aij]$ of order *n*. We can associate a number called the determinant of square matrix *A*, where $aij = (i, j)^{\text{th}}$ element of *A*.

Thus, the determinant is a number associated to a square matrix.

Hence, the correct answer is C.

Exercise 4.3

Question 1:

Find area of the triangle with vertices at the point given in each of the following:

(i) (1, 0), (6, 0), (4, 3) (ii) (2, 7), (1, 1), (10, 8)

(iii) (−2, −3), (3, 2), (−1, −8)

Answer

(i) The area of the triangle with vertices (1, 0), (6, 0), (4, 3) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 6 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[1(0-3)-0(6-4)+1(18-0)\right]$$

$$= \frac{1}{2}\left[-3+18\right] = \frac{15}{2} \text{ square units}$$

(ii) The area of the triangle with vertices (2, 7), (1, 1), (10, 8) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & 7 & 1 \\ 1 & 1 & 1 \\ 10 & 8 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(1-8)-7(1-10)+1(8-10)\right]$$

$$= \frac{1}{2}\left[2(-7)-7(-9)+1(-2)\right]$$

$$= \frac{1}{2}\left[-14+63-2\right] = \frac{1}{2}\left[-16+63\right]$$

$$= \frac{47}{2} \text{ square units}$$

(iii) The area of the triangle with vertices (−2, −3), (3, 2), (−1, −8)

is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}-2 & -3 & 1\\ 3 & 2 & 1\\ -1 & -8 & 1\end{vmatrix}$$

$$=\frac{1}{2}\left[-2(2+8)+3(3+1)+1(-24+2)\right]$$

$$=\frac{1}{2}\left[-2(10)+3(4)+1(-22)\right]$$

$$=\frac{1}{2}\left[-20+12-22\right]$$

$$=-\frac{30}{2}=-15$$

Hence, the area of the triangle is $|-15|=15\text{ square units}$.

Question 2:

Show that points

$A(a,b+c),B(b,c+a),C(c,a+b)$ are collinear

Answer

Area of ΔABC is given by the relation,

$$\Delta=\frac{1}{2}\begin{vmatrix}a & b+c & 1\\ b & c+a & 1\\ c & a+b & 1\end{vmatrix}$$

$$=\frac{1}{2}\begin{vmatrix}a & b+c & 1\\ b-a & a-b & 0\\ c-a & a-c & 0\end{vmatrix}\quad(\text{Applying } R_2\to R_2-R_1 \text{ and } R_3\to R_3-R_1)$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix}a & b+c & 1\\ -1 & 1 & 0\\ 1 & -1 & 0\end{vmatrix}$$

$$=\frac{1}{2}(a-b)(c-a)\begin{vmatrix}a & b+c & 1\\ -1 & 1 & 0\\ 0 & 0 & 0\end{vmatrix}\quad(\text{Applying } R_3\to R_3+R_2)$$

$$=0\qquad(\text{All elements of } R_3 \text{ are } 0)$$

Thus, the area of the triangle formed by points A, B, and C is zero.

Hence, the points A, B, and C are collinear.

Question 3:

Find values of k if area of triangle is 4 square units and vertices are

(i) $(k, 0)$, $(4, 0)$, $(0, 2)$ (ii) $(-2, 0)$, $(0, 4)$, $(0, k)$

Answer

We know that the area of a triangle whose vertices are (x_1, y_1), (x_2, y_2), and (x_3, y_3) is the absolute value of the determinant (Δ), where

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

It is given that the area of triangle is 4 square units.

$\therefore \Delta = \pm 4.$

(i) The area of the triangle with vertices $(k, 0)$, $(4, 0)$, $(0, 2)$ is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} k & 0 & 1 \\ 4 & 0 & 1 \\ 0 & 2 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[k(0-2) - 0(4-0) + 1(8-0)\right]$$

$$= \frac{1}{2}\left[-2k+8\right] = -k+4$$

$\therefore -k + 4 = \pm 4$

When $-k + 4 = -4$, $k = 8$.

When $-k + 4 = 4$, $k = 0$.

Hence, $k = 0, 8$.

(ii) The area of the triangle with vertices (−2, 0), (0, 4), (0, *k*) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 4 & 1 \\ 0 & k & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[-2(4-k)\right]$$

$$= k-4$$

$\therefore k - 4 = \pm 4$

When $k - 4 = -4$, $k = 0$.

When $k - 4 = 4$, $k = 8$.

Hence, $k = 0, 8$.

Question 4:

(i) Find equation of line joining (1, 2) and (3, 6) using determinants

(ii) Find equation of line joining (3, 1) and (9, 3) using determinants

Answer

(i) Let P (*x*, *y*) be any point on the line joining points A (1, 2) and B (3, 6). Then, the points A, B, and P are collinear. Therefore, the area of triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 1 & 2 & 1 \\ 3 & 6 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[1(6-y)-2(3-x)+1(3y-6x)\right]=0$$

$$\Rightarrow 6-y-6+2x+3y-6x=0$$

$$\Rightarrow 2y-4x=0$$

$$\Rightarrow y=2x$$

Hence, the equation of the line joining the given points is $y = 2x$.

(ii) Let P (*x*, *y*) be any point on the line joining points A (3, 1) and

B (9, 3). Then, the points A, B, and P are collinear. Therefore, the area of triangle ABP will be zero.

$$\therefore \frac{1}{2}\begin{vmatrix} 3 & 1 & 1 \\ 9 & 3 & 1 \\ x & y & 1 \end{vmatrix} = 0$$

$$\Rightarrow \frac{1}{2}\left[3(3-y)-1(9-x)+1(9y-3x)\right]=0$$

$$\Rightarrow 9-3y-9+x+9y-3x=0$$

$$\Rightarrow 6y-2x=0$$

$$\Rightarrow x-3y=0$$

Hence, the equation of the line joining the given points is $x - 3y = 0$.

Question 5:

If area of triangle is 35 square units with vertices (2, −6), (5, 4), and (k, 4). Then k is

A. 12 **B.** −2 **C.** −12, −2 **D.** 12, −2

Answer

Answer: D

The area of the triangle with vertices (2, −6), (5, 4), and (k, 4) is given by the relation,

$$\Delta = \frac{1}{2}\begin{vmatrix} 2 & -6 & 1 \\ 5 & 4 & 1 \\ k & 4 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[2(4-4)+6(5-k)+1(20-4k)\right]$$

$$= \frac{1}{2}\left[30-6k+20-4k\right]$$

$$= \frac{1}{2}\left[50-10k\right]$$

$$= 25-5k$$

It is given that the area of the triangle is ±35.

Therefore, we have:

$$\Rightarrow 25-5k=\pm 35$$

$$\Rightarrow 5(5-k)=\pm 35$$

$$\Rightarrow 5-k=\pm 7$$

When $5 - k = -7$, $k = 5 + 7 = 12$.

When $5 - k = 7$, $k = 5 - 7 = -2$.

Hence, $k = 12, -2$.

The correct answer is D.

Exercise 4.4

Question 1:

Write Minors and Cofactors of the elements of following determinants:

(i) $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$ (ii) $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$

Answer

(i) The given determinant is $\begin{vmatrix} 2 & -4 \\ 0 & 3 \end{vmatrix}$.

Minor of element a_{ij} is M_{ij}.

$\therefore M_{11}$ = minor of element $a_{11} = 3$

M_{12} = minor of element $a_{12} = 0$

M_{21} = minor of element $a_{21} = -4$

M_{22} = minor of element $a_{22} = 2$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (3) = 3$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (0) = 0$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (-4) = 4$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (2) = 2$

(ii) The given determinant is $\begin{vmatrix} a & c \\ b & d \end{vmatrix}$.

Minor of element a_{ij} is M_{ij}.

$\therefore M_{11}$ = minor of element $a_{11} = d$

M_{12} = minor of element $a_{12} = b$

M_{21} = minor of element $a_{21} = c$

M_{22} = minor of element $a_{22} = a$

Cofactor of a_{ij} is $A_{ij} = (-1)^{i+j} M_{ij}$.

$\therefore A_{11} = (-1)^{1+1} M_{11} = (-1)^2 (d) = d$

$A_{12} = (-1)^{1+2} M_{12} = (-1)^3 (b) = -b$

$A_{21} = (-1)^{2+1} M_{21} = (-1)^3 (c) = -c$

$A_{22} = (-1)^{2+2} M_{22} = (-1)^4 (a) = a$

Question 2:

(i) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ (ii) $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$

Answer

(i) The given determinant is $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$.

By the definition of minors and cofactors, we have:

M_{11} = minor of $a_{11} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

M_{12} = minor of $a_{12} = \begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

M_{13} = minor of a_{13} = $\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

M_{21} = minor of a_{21} = $\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = 0$

M_{22} = minor of a_{22} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

M_{23} = minor of a_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

M_{31} = minor of a_{31}= $\begin{vmatrix} 0 & 0 \\ 1 & 0 \end{vmatrix} = 0$

M_{32} = minor of a_{32} = $\begin{vmatrix} 1 & 0 \\ 0 & 0 \end{vmatrix} = 0$

M_{33} = minor of a_{33} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

A_{11} = cofactor of a_{11}= $(-1)^{1+1}$ M_{11} = 1

A_{12} = cofactor of a_{12} = $(-1)^{1+2}$ M_{12} = 0

A_{13} = cofactor of a_{13} = $(-1)^{1+3}$ M_{13} = 0

A_{21} = cofactor of a_{21} = $(-1)^{2+1}$ M_{21} = 0

A_{22} = cofactor of a_{22} = $(-1)^{2+2}$ M_{22} = 1

A_{23} = cofactor of a_{23} = $(-1)^{2+3}$ M_{23} = 0

A_{31} = cofactor of a_{31} = $(-1)^{3+1}$ M_{31} = 0

A_{32} = cofactor of a_{32} = $(-1)^{3+2}$ M_{32} = 0

A_{33} = cofactor of a_{33} = $(-1)^{3+3}$ M_{33} = 1

(ii) The given determinant is $\begin{vmatrix} 1 & 0 & 4 \\ 3 & 5 & -1 \\ 0 & 1 & 2 \end{vmatrix}$.

By definition of minors and cofactors, we have:

M_{11} = minor of a_{11}= $\begin{vmatrix} 5 & -1 \\ 1 & 2 \end{vmatrix} = 10 + 1 = 11$

M_{12} = minor of a_{12}= $\begin{vmatrix} 3 & -1 \\ 0 & 2 \end{vmatrix} = 6-0=6$

M_{13} = minor of a_{13} = $\begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3-0=3$

M_{21} = minor of a_{21} = $\begin{vmatrix} 0 & 4 \\ 1 & 2 \end{vmatrix} = 0-4=-4$

M_{22} = minor of a_{22} = $\begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix} = 2-0=2$

M_{23} = minor of a_{23} = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1-0=1$

M_{31} = minor of a_{31}= $\begin{vmatrix} 0 & 4 \\ 5 & -1 \end{vmatrix} = 0-20=-20$

M_{32} = minor of a_{32} = $\begin{vmatrix} 1 & 4 \\ 3 & -1 \end{vmatrix} = -1-12=-13$

M_{33} = minor of a_{33} = $\begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} = 5-0=5$

A_{11} = cofactor of a_{11}= $(-1)^{1+1}$ M_{11} = 11

A_{12} = cofactor of a_{12} = $(-1)^{1+2}$ M_{12} = −6

A_{13} = cofactor of a_{13} = $(-1)^{1+3}$ M_{13} = 3

A_{21} = cofactor of a_{21} = $(-1)^{2+1}$ M_{21} = 4

A_{22} = cofactor of a_{22} = $(-1)^{2+2}$ M_{22} = 2

A_{23} = cofactor of a_{23} = $(-1)^{2+3}$ M_{23} = −1

A_{31} = cofactor of a_{31} = $(-1)^{3+1}$ M_{31} = −20

A_{32} = cofactor of a_{32} = $(-1)^{3+2}$ M_{32} = 13

A_{33} = cofactor of a_{33} = $(-1)^{3+3}$ M_{33} = 5

Question 3:

Using Cofactors of elements of second row, evaluate $\Delta = \begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

Answer

The given determinant is $\begin{vmatrix} 5 & 3 & 8 \\ 2 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$.

We have:

$M_{21} = \begin{vmatrix} 3 & 8 \\ 2 & 3 \end{vmatrix} = 9 - 16 = -7$

$\therefore A_{21}$ = cofactor of $a_{21} = (-1)^{2+1} M_{21} = 7$

$M_{22} = \begin{vmatrix} 5 & 8 \\ 1 & 3 \end{vmatrix} = 15 - 8 = 7$

$\therefore A_{22}$ = cofactor of $a_{22} = (-1)^{2+2} M_{22} = 7$

$M_{23} = \begin{vmatrix} 5 & 3 \\ 1 & 2 \end{vmatrix} = 10 - 3 = 7$

$\therefore A_{23}$ = cofactor of $a_{23} = (-1)^{2+3} M_{23} = -7$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$\therefore \Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23} = 2(7) + 0(7) + 1(-7) = 14 - 7 = 7$

Question 4:

Using Cofactors of elements of third column, evaluate $\Delta = \begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$

Answer

The given determinant is $\begin{vmatrix} 1 & x & yz \\ 1 & y & zx \\ 1 & z & xy \end{vmatrix}$.

We have:

$M_{13} = \begin{vmatrix} 1 & y \\ 1 & z \end{vmatrix} = z - y$

$M_{23} = \begin{vmatrix} 1 & x \\ 1 & z \end{vmatrix} = z - x$

$M_{33} = \begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix} = y - x$

$\therefore A_{13} = \text{cofactor of } a_{13} = (-1)^{1+3} M_{13} = (z - y)$

$A_{23} = \text{cofactor of } a_{23} = (-1)^{2+3} M_{23} = -(z - x) = (x - z)$

$A_{33} = \text{cofactor of } a_{33} = (-1)^{3+3} M_{33} = (y - x)$

We know that Δ is equal to the sum of the product of the elements of the second row with their corresponding cofactors.

$$\therefore \Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$$
$$= yz(z-y) + zx(x-z) + xy(y-x)$$
$$= yz^2 - y^2z + x^2z - xz^2 + xy^2 - x^2y$$
$$= (x^2z - y^2z) + (yz^2 - xz^2) + (xy^2 - x^2y)$$
$$= z(x^2 - y^2) + z^2(y-x) + xy(y-x)$$
$$= z(x-y)(x+y) + z^2(y-x) + xy(y-x)$$
$$= (x-y)\left[zx + zy - z^2 - xy\right]$$
$$= (x-y)\left[z(x-z) + y(z-x)\right]$$
$$= (x-y)(z-x)\left[-z+y\right]$$
$$= (x-y)(y-z)(z-x)$$

Hence, $\Delta = (x-y)(y-z)(z-x)$.

Question 5:

For the matrices *A* and *B*, verify that (*AB*)′ = $B'A'$ where

(i) $A = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, B = \begin{bmatrix} -1 & 2 & 1 \end{bmatrix}$

(ii) $A = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}, B = \begin{bmatrix} 1 & 5 & 7 \end{bmatrix}$

Answer

(i)

$$AB = \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}\begin{bmatrix} -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 4 & -8 & -4 \\ -3 & 6 & 3 \end{bmatrix}$$

$$\therefore (AB)' = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Now, $A' = \begin{bmatrix} 1 & -4 & 3 \end{bmatrix}$, $B' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$

$$\therefore B'A' = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & -4 & 3 \end{bmatrix} = \begin{bmatrix} -1 & 4 & -3 \\ 2 & -8 & 6 \\ 1 & -4 & 3 \end{bmatrix}$$

Hence, we have verified that $(AB)' = B'A'$.

(ii)

$$AB=\begin{bmatrix}0\\1\\2\end{bmatrix}\begin{bmatrix}1 & 5 & 7\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\1 & 5 & 7\\2 & 10 & 14\end{bmatrix}$$

$$\therefore (AB)'=\begin{bmatrix}0 & 1 & 2\\0 & 5 & 10\\0 & 7 & 14\end{bmatrix}$$

Now, $A'=\begin{bmatrix}0 & 1 & 2\end{bmatrix}, B'=\begin{bmatrix}1\\5\\7\end{bmatrix}$

$$\therefore B'A'=\begin{bmatrix}1\\5\\7\end{bmatrix}\begin{bmatrix}0 & 1 & 2\end{bmatrix}=\begin{bmatrix}0 & 1 & 2\\0 & 5 & 10\\0 & 7 & 14\end{bmatrix}$$

Hence, we have verified that $(AB)'=B'A'$.

Exercise 4.5

Question 1:

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

We have,

$$A_{11} = 4,\ A_{12} = -3,\ A_{21} = -2,\ A_{22} = 1$$

$$\therefore adjA = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Question 2:

Find adjoint of each of the matrices.

$$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 3 & 5 \\ -2 & 0 & 1 \end{bmatrix}.$$

We have,

$$A_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 - 0 = 3$$

$$A_{12} = -\begin{vmatrix} 2 & 5 \\ -2 & 1 \end{vmatrix} = -(2+10) = -12$$

$$A_{13} = \begin{vmatrix} 2 & 3 \\ -2 & 0 \end{vmatrix} = 0 + 6 = 6$$

$$A_{21} = -\begin{vmatrix} -1 & 2 \\ 0 & 1 \end{vmatrix} = -(-1-0) = 1$$

$$A_{22} = \begin{vmatrix} 1 & 2 \\ -2 & 1 \end{vmatrix} = 1+4 = 5$$

$$A_{23} = -\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix} = -(0-2) = 2$$

$$A_{31} = \begin{vmatrix} -1 & 2 \\ 3 & 5 \end{vmatrix} = -5-6 = -11$$

$$A_{32} = -\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = -(5-4) = -1$$

$$A_{33} = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3+2 = 5$$

Hence, $adjA = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix} = \begin{bmatrix} 3 & 1 & -11 \\ -12 & 5 & -1 \\ 6 & 2 & 5 \end{bmatrix}$.

Question 3:

Verify *A* (*adj A*) = (*adj A*) *A* = $|A| I$.

$$\begin{bmatrix} 2 & 3 \\ -4 & -6 \end{bmatrix}$$

Answer

$$A=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$$

we have,

$$|A|=-12-(-12)=-12+12=0$$

$$\therefore |A|I=0\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Now,

$$A_{11}=-6, A_{12}=4, A_{21}=-3, A_{22}=2$$

$$\therefore adjA=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}$$

Now,

$$A(adjA)=\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}$$

$$=\begin{bmatrix}-12+12 & -6+6\\ 24-24 & 12-12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Also, $(adjA)A=\begin{bmatrix}-6 & -3\\ 4 & 2\end{bmatrix}\begin{bmatrix}2 & 3\\ -4 & -6\end{bmatrix}$

$$=\begin{bmatrix}-12+12 & -18+18\\ 8-8 & 12-12\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, $A(adjA)=(adjA)A=|A|I.$

Question 4:

Verify *A* (*adj A*) = (*adj A*) *A* = $|A|I$.

$$\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

Answer

$$A=\begin{bmatrix}1 & -1 & 2\\ 3 & 0 & -2\\ 1 & 0 & 3\end{bmatrix}$$

$$|A|=1(0-0)+1(9+2)+2(0-0)=11$$

$$\therefore |A|I=11\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}11 & 0 & 0\\ 0 & 11 & 0\\ 0 & 0 & 11\end{bmatrix}$$

Now,

$$A_{11}=0, A_{12}=-(9+2)=-11, A_{13}=0$$

$$A_{21}=-(-3-0)=3, A_{22}=3-2=1, A_{23}=-(0+1)=-1$$

$$A_{31}=2-0=2, A_{32}=-(-2-6)=8, A_{33}=0+3=3$$

$$\therefore adjA = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

Now,

$$A(adjA) = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+11+0 & 3-1-2 & 2-8+6 \\ 0+0+0 & 9+0+2 & 6+0-6 \\ 0+0+0 & 3+0-3 & 2+0+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Also,

$$(adjA)\cdot A = \begin{bmatrix} 0 & 3 & 2 \\ -11 & 1 & 8 \\ 0 & -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & -2 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+9+2 & 0+0+0 & 0-6+6 \\ -11+3+8 & 11+0+0 & -22-2+24 \\ 0-3+3 & 0+0+0 & 0+2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

Hence, $A(adjA) = (adjA)A = |A|I$.

Question 6:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix}$$

Answer

Let $A=\begin{bmatrix}-1 & 5\\ -3 & 2\end{bmatrix}$.

we have,

$|A|=-2+15=13$

Now,

$A_{11}=2, A_{12}=3, A_{21}=-5, A_{22}=-1$

$\therefore adjA=\begin{bmatrix}2 & -5\\ 3 & -1\end{bmatrix}$

$\therefore A^{-1}=\frac{1}{|A|}adjA=\frac{1}{13}\begin{bmatrix}2 & -5\\ 3 & -1\end{bmatrix}$

Question 7:

Find the inverse of each of the matrices (if it exists).

$\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix}$

Answer

Let $A=\begin{bmatrix}1 & 2 & 3\\ 0 & 2 & 4\\ 0 & 0 & 5\end{bmatrix}$.

We have,

$|A|=1(10-0)-2(0-0)+3(0-0)=10$

Now,

$A_{11}=10-0=10, A_{12}=-(0-0)=0, A_{13}=0-0=0$

$A_{21}=-(10-0)=-10, A_{22}=5-0=5, A_{23}=-(0-0)=0$

$A_{31}=8-6=2, A_{32}=-(4-0)=-4, A_{33}=2-0=2$

$$\therefore adjA = \begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{|A|} adjA = \frac{1}{10}\begin{bmatrix} 10 & -10 & 2 \\ 0 & 5 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

Question 8:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 3 & 0 \\ 5 & 2 & -1 \end{bmatrix}.$$

We have,

$$|A| = 1(-3-0) - 0 + 0 = -3$$

Now,

$$A_{11} = -3-0 = -3, A_{12} = -(-3-0) = 3, A_{13} = 6-15 = -9$$

$$A_{21} = -(0-0) = 0, A_{22} = -1-0 = -1, A_{23} = -(2-0) = -2$$

$$A_{31} = 0-0 = 0, A_{32} = -(0-0) = 0, A_{33} = 3-0 = 3$$

$$\therefore adjA = \begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

$$\therefore \ A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3}\begin{bmatrix} -3 & 0 & 0 \\ 3 & -1 & 0 \\ -9 & -2 & 3 \end{bmatrix}$$

Question 9:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}$$

Answer

$$\text{Let } A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & -1 & 0 \\ -7 & 2 & 1 \end{bmatrix}.$$

We have,

$$\begin{aligned} |A| &= 2(-1-0)-1(4-0)+3(8-7) \\ &= 2(-1)-1(4)+3(1) \\ &= -2-4+3 \\ &= -3 \end{aligned}$$

Now,

$$A_{11} = -1-0 = -1, A_{12} = -(4-0) = -4, A_{13} = 8-7 = 1$$

$$A_{21} = -(1-6) = 5, A_{22} = 2+21 = 23, A_{23} = -(4+7) = -11$$

$$A_{31} = 0+3 = 3, A_{32} = -(0-12) = 12, A_{33} = -2-4 = -6$$

$$\therefore adjA = \begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -\frac{1}{3}\begin{bmatrix} -1 & 5 & 3 \\ -4 & 23 & 12 \\ 1 & -11 & -6 \end{bmatrix}$$

Question 10:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}.$$

Answer

Let $A = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$.

By expanding along C_1, we have:

$|A| = 1(8-6) - 0 + 3(3-4) = 2 - 3 = -1$

Now,

$A_{11} = 8-6 = 2, A_{12} = -(0+9) = -9, A_{13} = 0-6 = -6$

$A_{21} = -(-4+4) = 0, A_{22} = 4-6 = -2, A_{23} = -(-2+3) = -1$

$A_{31} = 3-4 = -1, A_{32} = -(-3-0) = 3, A_{33} = 2-0 = 2$

$$\therefore adjA = \begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = -\begin{bmatrix} 2 & 0 & -1 \\ -9 & -2 & 3 \\ -6 & -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 1 \\ 9 & 2 & -3 \\ 6 & 1 & -2 \end{bmatrix}$$

Question 11:

Find the inverse of each of the matrices (if it exists).

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & \sin\alpha & -\cos\alpha \end{bmatrix}$$

Answer

$$\text{Let } A=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}.$$

We have,

$$|A|=1\left(-\cos^2\alpha-\sin^2\alpha\right)=-\left(\cos^2\alpha+\sin^2\alpha\right)=-1$$

Now,

$$A_{11}=-\cos^2\alpha-\sin^2\alpha=-1, A_{12}=0, A_{13}=0$$
$$A_{21}=0, A_{22}=-\cos\alpha, A_{23}=-\sin\alpha$$
$$A_{31}=0, A_{32}=-\sin\alpha, A_{33}=\cos\alpha$$

$$\therefore adjA=\begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}\cdot adjA=-\begin{bmatrix}-1 & 0 & 0\\ 0 & -\cos\alpha & -\sin\alpha\\ 0 & -\sin\alpha & \cos\alpha\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\alpha & \sin\alpha\\ 0 & \sin\alpha & -\cos\alpha\end{bmatrix}$$

Question 12:

Let $A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$. Verify that $(AB)^{-1}=B^{-1}A^{-1}$

Answer

$$\text{Let } A=\begin{bmatrix}3 & 7\\ 2 & 5\end{bmatrix}.$$

We have,

$$|A|=15-14=1$$

Now,

$$A_{11}=5, A_{12}=-2, A_{21}=-7, A_{22}=3$$

$$\therefore adjA=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$\therefore A^{-1}=\frac{1}{|A|}\cdot adjA=\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

Now, let $B=\begin{bmatrix}6 & 8\\ 7 & 9\end{bmatrix}$.

We have,

$|B| = 54-56=-2$

$\therefore adjB=\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}$

$$\therefore B^{-1}=\frac{1}{|B|}adjB=-\frac{1}{2}\begin{bmatrix}9 & -8\\ -7 & 6\end{bmatrix}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}$$

Now,

$$B^{-1}A^{-1}=\begin{bmatrix}-\frac{9}{2} & 4\\ \frac{7}{2} & -3\end{bmatrix}\begin{bmatrix}5 & -7\\ -2 & 3\end{bmatrix}$$

$$=\begin{bmatrix}-\frac{45}{2}-8 & \frac{63}{2}+12\\ \frac{35}{2}+6 & -\frac{49}{2}-9\end{bmatrix}=\begin{bmatrix}-\frac{61}{2} & \frac{87}{2}\\ \frac{47}{2} & -\frac{67}{2}\end{bmatrix} \qquad ...(1)$$

Then,

$$AB = \begin{bmatrix} 3 & 7 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} 6 & 8 \\ 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 18+49 & 24+63 \\ 12+35 & 16+45 \end{bmatrix}$$

$$= \begin{bmatrix} 67 & 87 \\ 47 & 61 \end{bmatrix}$$

Therefore, we have $|AB| = 67 \times 61 - 87 \times 47 = 4087 - 4089 = -2.$

Also,

$$adj(AB) = \begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$\therefore (AB)^{-1} = \frac{1}{|AB|} adj(AB) = -\frac{1}{2}\begin{bmatrix} 61 & -87 \\ -47 & 67 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{61}{2} & \frac{87}{2} \\ \frac{47}{2} & -\frac{67}{2} \end{bmatrix} \quad ...(2)$$

From (1) and (2), we have:

$(AB)^{-1} = B^{-1}A^{-1}$

Hence, the given result is proved.

Question 13:

If $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, show that $A^2 - 5A + 7I = O$. Hence find A^{-1}.

Answer

$$A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$$

$$A^2=A\cdot A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}=\begin{bmatrix}9-1 & 3+2\\ -3-2 & -1+4\end{bmatrix}=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}$$

$$\therefore A^2-5A+7I$$

$$=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-5\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}+7\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}8 & 5\\ -5 & 3\end{bmatrix}-\begin{bmatrix}15 & 5\\ -5 & 10\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}$$

$$=\begin{bmatrix}-7 & 0\\ 0 & -7\end{bmatrix}+\begin{bmatrix}7 & 0\\ 0 & 7\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$$

Hence, $A^2-5A+7I=O$.

$$\therefore A\cdot A-5A=-7I$$

$$\Rightarrow A\cdot A\left(A^{-1}\right)-5AA^{-1}=-7IA^{-1} \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right)-5I=-7A^{-1}$$

$$\Rightarrow AI-5I=-7A^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{7}(A-5I)$$

$$\Rightarrow A^{-1}=\frac{1}{7}(5I-A)$$

$$=\frac{1}{7}\left(\begin{bmatrix}5 & 0\\ 0 & 5\end{bmatrix}-\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}\right)=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{7}\begin{bmatrix}2 & -1\\ 1 & 3\end{bmatrix}$$

Question 14:

For the matrix $A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$, find the numbers *a* and *b* such that $A^2 + aA + bI = O$.

Answer

$$A=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}$$

$$\therefore A^2=\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}=\begin{bmatrix}9+2 & 6+2\\ 3+1 & 2+1\end{bmatrix}=\begin{bmatrix}11 & 8\\ 4 & 3\end{bmatrix}$$

Now,

$$A^2+aA+bI=O$$

$$\Rightarrow (AA)A^{-1}+aAA^{-1}+bIA^{-1}=O \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$$

$$\Rightarrow A\left(AA^{-1}\right)+aI+b\left(IA^{-1}\right)=O$$

$$\Rightarrow AI+aI+bA^{-1}=O$$

$$\Rightarrow A+aI=-bA^{-1}$$

$$\Rightarrow A^{-1}=-\frac{1}{b}(A+aI)$$

Now,

$$A^{-1}=\frac{1}{|A|}adjA=\frac{1}{1}\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}$$

We have:

$$\begin{bmatrix}1 & -2\\ -1 & 3\end{bmatrix}=-\frac{1}{b}\left(\begin{bmatrix}3 & 2\\ 1 & 1\end{bmatrix}+\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}\right)=-\frac{1}{b}\begin{bmatrix}3+a & 2\\ 1 & 1+a\end{bmatrix}=\begin{bmatrix}\frac{-3-a}{b} & -\frac{2}{b}\\ -\frac{1}{b} & \frac{-1-a}{b}\end{bmatrix}$$

Comparing the corresponding elements of the two matrices, we have:

$$-\frac{1}{b}=-1\Rightarrow b=1$$

$$\frac{-3-a}{b}=1\Rightarrow -3-a=1\Rightarrow a=-4$$

Hence, −4 and 1 are the required values of *a* and *b* respectively.

Question 15:

For the matrix $A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$ show that $A^3 - 6A^2 + 5A + 11\ I = O$. Hence, find $A^{-1.}$

Answer

$$A=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$A^2=\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}1+1+2 & 1+2-1 & 1-3+3\\ 1+2-6 & 1+4+3 & 1-6-9\\ 2-1+6 & 2-2-3 & 2+3+9\end{bmatrix}=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}4 & 2 & 1\\ -3 & 8 & -14\\ 7 & -3 & 14\end{bmatrix}\begin{bmatrix}1 & 1 & 1\\ 1 & 2 & -3\\ 2 & -1 & 3\end{bmatrix}$$

$$=\begin{bmatrix}4+2+2 & 4+4-1 & 4-6+3\\ -3+8-28 & -3+16+14 & -3-24-42\\ 7-3+28 & 7-6-14 & 7+9+42\end{bmatrix}$$

$$=\begin{bmatrix}8 & 7 & 1\\ -23 & 27 & -69\\ 32 & -13 & 58\end{bmatrix}$$

$\therefore A^3 - 6A^2 + 5A + 11I$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - 6\begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} + 5\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 11\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 7 & 1 \\ -23 & 27 & -69 \\ 32 & -13 & 58 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} + \begin{bmatrix} 5 & 5 & 5 \\ 5 & 10 & -15 \\ 10 & -5 & 15 \end{bmatrix} + \begin{bmatrix} 11 & 0 & 0 \\ 0 & 11 & 0 \\ 0 & 0 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix} - \begin{bmatrix} 24 & 12 & 6 \\ -18 & 48 & -84 \\ 42 & -18 & 84 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

Thus, $A^3 - 6A^2 + 5A + 11I = O$.

Now,

$A^3 - 6A^2 + 5A + 11I = O$

$\Rightarrow (AAA)A^{-1} - 6(AA)A^{-1} + 5AA^{-1} + 11IA^{-1} = 0 \qquad \left[\text{Post-multiplying by } A^{-1} \text{ as } |A| \neq 0\right]$

$\Rightarrow AA(AA^{-1}) - 6A(AA^{-1}) + 5(AA^{-1}) = -11(IA^{-1})$

$\Rightarrow A^2 - 6A + 5I = -11A^{-1}$

$\Rightarrow A^{-1} = -\frac{1}{11}(A^2 - 6A + 5I) \qquad \ldots(1)$

Now,

$A^2 - 6A + 5I$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - 6\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & -3 \\ 2 & -1 & 3 \end{bmatrix} + 5\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 2 & 1 \\ -3 & 8 & -14 \\ 7 & -3 & 14 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 2 & 1 \\ -3 & 13 & -14 \\ 7 & -3 & 19 \end{bmatrix} - \begin{bmatrix} 6 & 6 & 6 \\ 6 & 12 & -18 \\ 12 & -6 & 18 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix}$$

From equation (1), we have:

$$A^{-1} = -\frac{1}{11}\begin{bmatrix} 3 & -4 & -5 \\ -9 & 1 & 4 \\ -5 & 3 & 1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -3 & 4 & 5 \\ 9 & -1 & -4 \\ 5 & -3 & -1 \end{bmatrix}$$

Question 16:

If $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ verify that $A^3 - 6A^2 + 9A - 4I = O$ and hence find A^{-1}

Answer

$$A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$A^2=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}4+1+1 & -2-2-1 & 2+1+2\\ -2-2-1 & 1+4+1 & -1-2-2\\ 2+1+2 & -1-2-2 & 1+1+4\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$$

$$A^3=A^2A=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$

$$=\begin{bmatrix}12+5+5 & -6-10-5 & 6+5+10\\ -10-6-5 & 5+12+5 & -5-6-10\\ 10+5+6 & -5-10-6 & 5+5+12\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$$

Now,

$A^3-6A^2+9A-4I$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}+9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}-4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}-\begin{bmatrix}36 & -30 & 30\\ -30 & 36 & -30\\ 30 & -30 & 36\end{bmatrix}+\begin{bmatrix}18 & -9 & 9\\ -9 & 18 & -9\\ 9 & -9 & 18\end{bmatrix}-\begin{bmatrix}4 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 4\end{bmatrix}$$

$$=\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}-\begin{bmatrix}40 & -30 & 30\\ -30 & 40 & -30\\ 30 & -30 & 40\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$\therefore A^3-6A^2+9A-4I=O$

Now,

$A^3-6A^2+9A-4I=O$

$\Rightarrow (AAA)A^{-1}-6(AA)A^{-1}+9AA^{-1}-4IA^{-1}=O$ $\qquad$ $\left[\text{Post-multiplying by } A^{-1} \text{ as } |A|\neq 0\right]$

$\Rightarrow AA\left(AA^{-1}\right)-6A\left(AA^{-1}\right)+9\left(AA^{-1}\right)=4\left(IA^{-1}\right)$

$\Rightarrow AAI-6AI+9I=4A^{-1}$

$\Rightarrow A^2-6A+9I=4A^{-1}$

$\Rightarrow A^{-1}=\frac{1}{4}\left(A^2-6A+9I\right)$ $\qquad \ldots(1)$

$A^2-6A+9I$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-6\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}+9\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$=\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}-\begin{bmatrix}12 & -6 & 6\\ -6 & 12 & -6\\ 6 & -6 & 12\end{bmatrix}+\begin{bmatrix}9 & 0 & 0\\ 0 & 9 & 0\\ 0 & 0 & 9\end{bmatrix}$$

$$=\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

From equation (1), we have:

$$A^{-1}=\frac{1}{4}\begin{bmatrix}3 & 1 & -1\\ 1 & 3 & 1\\ -1 & 1 & 3\end{bmatrix}$$

Question 17:

Let A be a nonsingular square matrix of order 3 × 3. Then $|adjA|$ is equal to

A. $|A|$ **B.** $|A|^2$ **C.** $|A|^3$ **D.** $3|A|$

Answer **B**

We know that,

$$(adjA)A=|A|I=\begin{bmatrix}|A| & 0 & 0\\ 0 & |A| & 0\\ 0 & 0 & |A|\end{bmatrix}$$

$$\Rightarrow |(adjA)A|=\begin{vmatrix}|A| & 0 & 0\\ 0 & |A| & 0\\ 0 & 0 & |A|\end{vmatrix}$$

$$\Rightarrow |adjA||A|=|A|^3\begin{vmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{vmatrix}=|A|^3(I)$$

$$\therefore |adjA|=|A|^2$$

Hence, the correct answer is B.

Question 18:

If A is an invertible matrix of order 2, then det (A^{-1}) is equal to

A. det (A) **B.** $\frac{1}{\det(A)}$ **C.** 1 **D.** 0

Answer

Since A is an invertible matrix, A^{-1} exists and $A^{-1}=\frac{1}{|A|}adjA.$

As matrix A is of order 2, let $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then, $|A|=ad-bc$ and $adjA=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Now,

$$A^{-1}=\frac{1}{|A|}adjA=\begin{bmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{bmatrix}$$

$$\therefore \left|A^{-1}\right|=\begin{vmatrix} \frac{d}{|A|} & \frac{-b}{|A|} \\ \frac{-c}{|A|} & \frac{a}{|A|} \end{vmatrix}=\frac{1}{|A|^2}\begin{vmatrix} d & -b \\ -c & a \end{vmatrix}=\frac{1}{|A|^2}(ad-bc)=\frac{1}{|A|^2}.|A|=\frac{1}{|A|}$$

$$\therefore \det\left(A^{-1}\right)=\frac{1}{\det(A)}$$

Hence, the correct answer is B.

Exercise 4.6

Question 1:

Examine the consistency of the system of equations.

$x + 2y = 2$

$2x + 3y = 3$

Answer

The given system of equations is:

$x + 2y = 2$

$2x + 3y = 3$

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 1(3) - 2(2) = 3 - 4 = -1 \neq 0$$

∴ A is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 2:

Examine the consistency of the system of equations.

$2x - y = 5$

$x + y = 4$

Answer

The given system of equations is:

$2x - y = 5$

$x + y = 4$

The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & -1\\ 1 & 1\end{bmatrix},\ X=\begin{bmatrix}x\\ z\end{bmatrix}\text{ and }B=\begin{bmatrix}5\\ 4\end{bmatrix}.$$

Now,

$$|A|=2(1)-(-1)(1)=2+1=3\neq 0$$

∴ *A* is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 3:

Examine the consistency of the system of equations.

$x + 3y = 5$

$2x + 6y = 8$

Answer

The given system of equations is:

$x + 3y = 5$

$2x + 6y = 8$

The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}1 & 3\\ 2 & 6\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix}\text{ and }B=\begin{bmatrix}5\\ 8\end{bmatrix}.$$

Now,

$$|A|=1(6)-3(2)=6-6=0$$

∴ *A* is a singular matrix.

Now,

$$(adjA)=\begin{bmatrix}6 & -3\\ -2 & 1\end{bmatrix}$$

$$(adjA)B=\begin{bmatrix}6 & -3\\ -2 & 1\end{bmatrix}\begin{bmatrix}5\\ 8\end{bmatrix}=\begin{bmatrix}30-24\\ -10+8\end{bmatrix}=\begin{bmatrix}6\\ -2\end{bmatrix}\neq O$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent.

Question 4:

Examine the consistency of the system of equations.

$x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

Answer

The given system of equations is:

$x + y + z = 1$

$2x + 3y + 2z = 2$

$ax + ay + 2az = 4$

This system of equations can be written in the form $AX = B$, where

$$A=\begin{bmatrix}1 & 1 & 1\\ 2 & 3 & 2\\ a & a & 2a\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\text{ and }B=\begin{bmatrix}1\\ 2\\ 4\end{bmatrix}.$$

Now,

$$|A|=1(6a-2a)-1(4a-2a)+1(2a-3a)$$
$$=4a-2a-a=4a-3a=a\neq 0$$

$\therefore$ A is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 5:

Examine the consistency of the system of equations.

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

Answer

The given system of equations is:

$3x - y - 2z = 2$

$2y - z = -1$

$3x - 5y = 3$

This system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}3 & -1 & -2\\ 0 & 2 & -1\\ 3 & -5 & 0\end{bmatrix},\ X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}\text{ and } B=\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}.$$

Now,

$$|A| = 3(0-5)-0+3(1+4) = -15+15 = 0$$

$\therefore$ A is a singular matrix.

Now,

$$(adjA)=\begin{bmatrix}-5 & 10 & 5\\ -3 & 6 & 3\\ -6 & 12 & 6\end{bmatrix}$$

$$\therefore (adjA)B=\begin{bmatrix}-5 & 10 & 5\\ -3 & 6 & 3\\ -6 & 12 & 6\end{bmatrix}\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}=\begin{bmatrix}-10-10+15\\ -6-6+9\\ -12-12+18\end{bmatrix}=\begin{bmatrix}-5\\ -3\\ -6\end{bmatrix}\neq O$$

Thus, the solution of the given system of equations does not exist. Hence, the system of equations is inconsistent.

Question 6:

Examine the consistency of the system of equations.

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

Answer

The given system of equations is:

$5x - y + 4z = 5$

$2x + 3y + 5z = 2$

$5x - 2y + 6z = -1$

This system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 5 & -1 & 4 \\ 2 & 3 & 5 \\ 5 & -2 & 6 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}.$$

Now,

$$\begin{aligned} |A| &= 5(18+10)+1(12-25)+4(-4-15) \\ &= 5(28)+1(-13)+4(-19) \\ &= 140-13-76 \\ &= 51 \neq 0 \end{aligned}$$

∴ A is non-singular.

Therefore, A^{-1} exists.

Hence, the given system of equations is consistent.

Question 7:

Solve system of linear equations, using matrix method.

$5x + 2y = 4$

$7x + 3y = 5$

Answer

The given system of equations can be written in the form of *AX* = *B*, where

$$A=\begin{bmatrix}5 & 2\\ 7 & 3\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 5\end{bmatrix}.$$

Now, $|A|=15-14=1\neq 0.$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)$$

$$\therefore A^{-1}=\begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}$$

$$\therefore X=A^{-1}B=\begin{bmatrix}3 & -2\\ -7 & 5\end{bmatrix}\begin{bmatrix}4\\ 5\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}12-10\\ -28+25\end{bmatrix}=\begin{bmatrix}2\\ -3\end{bmatrix}$$

Hence, $x=2$ and $y=-3$.

Question 8:

Solve system of linear equations, using matrix method.

$2x-y=-2$

$3x+4y=3$

Answer

The given system of equations can be written in the form of *AX* = *B*, where

$$A=\begin{bmatrix}2 & -1\\ 3 & 4\end{bmatrix},\ X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}-2\\ 3\end{bmatrix}.$$

Now,

$|A|=8+3=11\neq 0$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{11}\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -8+3 \\ 6+6 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} -5 \\ 12 \end{bmatrix} = \begin{bmatrix} -\frac{5}{11} \\ \frac{12}{11} \end{bmatrix}$$

Hence, $x = \frac{-5}{11}$ and $y = \frac{12}{11}$.

Question 9:

Solve system of linear equations, using matrix method.

$4x - 3y = 3$

$3x - 5y = 7$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 4 & -3 \\ 3 & -5 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix} \text{and } B = \begin{bmatrix} 3 \\ 7 \end{bmatrix}.$$

Now,

$$|A| = -20 + 9 = -11 \neq 0$$

Thus, *A* is non-singular. Therefore, its inverse exists.

Now,

$$A^{-1}=\frac{1}{|A|}(adjA)=-\frac{1}{11}\begin{bmatrix}-5 & 3\\ -3 & 4\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}$$

$$\therefore X=A^{-1}B=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\ y\end{bmatrix}=\frac{1}{11}\begin{bmatrix}5 & -3\\ 3 & -4\end{bmatrix}\begin{bmatrix}3\\ 7\end{bmatrix}=\frac{1}{11}\begin{bmatrix}15-21\\ 9-28\end{bmatrix}=\frac{1}{11}\begin{bmatrix}-6\\ -19\end{bmatrix}=\begin{bmatrix}-\frac{6}{11}\\ -\frac{19}{11}\end{bmatrix}$$

Hence, $x=\frac{-6}{11}$ and $y=\frac{-19}{11}$.

Question 10:

Solve system of linear equations, using matrix method.

$5x + 2y = 3$

$3x + 2y = 5$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}5 & 2\\ 3 & 2\end{bmatrix}, X=\begin{bmatrix}x\\ y\end{bmatrix} \text{ and } B=\begin{bmatrix}3\\ 5\end{bmatrix}.$$

Now,

$$|A|=10-6=4\neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

Question 11:

Solve system of linear equations, using matrix method.

$$2x+y+z=1$$

$$x-2y-z=\frac{3}{2}$$

$$3y-5z=9$$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & 1 & 1\\1 & -2 & -1\\0 & 3 & -5\end{bmatrix},\ X=\begin{bmatrix}x\\y\\z\end{bmatrix} \text{ and } B=\begin{bmatrix}1\\\frac{3}{2}\\9\end{bmatrix}.$$

Now,

$$|A|=2(10+3)-1(-5-3)+0=2(13)-1(-8)=26+8=34\neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

Now, $A_{11}=13, A_{12}=5, A_{13}=3$

$A_{21}=8, A_{22}=-10, A_{23}=-6$

$A_{31}=1, A_{32}=3, A_{33}=-5$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\5 & -10 & 3\\3 & -6 & -5\end{bmatrix}$$

$$\therefore\ X=A^{-1}B=\frac{1}{34}\begin{bmatrix}13 & 8 & 1\\5 & -10 & 3\\3 & -6 & -5\end{bmatrix}\begin{bmatrix}1\\\frac{3}{2}\\9\end{bmatrix}$$

$$\Rightarrow\begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{34}\begin{bmatrix}13+12+9\\5-15+27\\3-9-45\end{bmatrix}$$

$$=\frac{1}{34}\begin{bmatrix}34\\17\\-51\end{bmatrix}=\begin{bmatrix}1\\\frac{1}{2}\\-\frac{3}{2}\end{bmatrix}$$

Hence, $x=1, y=\frac{1}{2}$, and $z=-\frac{3}{2}$.

Question 12:

Solve system of linear equations, using matrix method.

$x - y + z = 4$

$2x + y - 3z = 0$

$x + y + z = 2$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

Now,

$$|A| = 1(1+3)+1(2+3)+1(2-1) = 4+5+1 = 10 \neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

Now, $A_{11} = 4, A_{12} = -5, A_{13} = 1$

$A_{21} = 2, A_{22} = 0, A_{23} = -2$

$A_{31} = 2, A_{32} = 5, A_{33} = 3$

$$\therefore\ A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{10}\begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & 5 \\ 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 16+0+4 \\ -20+0+10 \\ 4+0+6 \end{bmatrix}$$

$$= \frac{1}{10}\begin{bmatrix} 20 \\ -10 \\ 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Hence, $x = 2, y = -1,$ and $z = 1$.

Question 13:

Solve system of linear equations, using matrix method.

$2x + 3y + 3z = 5$

$x - 2y + z = -4$

$3x - y - 2z = 3$

Answer

The given system of equations can be written in the form $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & 3 \\ 1 & -2 & 1 \\ 3 & -1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{and } B = \begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}.$$

Now,

$$|A| = 2(4+1) - 3(-2-3) + 3(-1+6) = 2(5) - 3(-5) + 3(5) = 10 + 15 + 15 = 40 \neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

Now, $A_{11} = 5, A_{12} = 5, A_{13} = 5$

$A_{21} = 3, A_{22} = -13, A_{23} = 11$

$A_{31} = 9, A_{32} = 1, A_{33} = -7$

$$\therefore A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{40}\begin{bmatrix} 5 & 3 & 9 \\ 5 & -13 & 1 \\ 5 & 11 & -7 \end{bmatrix}\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{40}\begin{bmatrix} 25-12+27 \\ 25+52+3 \\ 25-44-21 \end{bmatrix}$$

$$= \frac{1}{40}\begin{bmatrix} 40 \\ 80 \\ -40 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

Hence, $x = 1, y = 2, \text{and } z = -1.$

Question 14:

Solve system of linear equations, using matrix method.

$x - y + 2z = 7$

$3x + 4y - 5z = -5$

$2x - y + 3z = 12$

Answer

The given system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 4 & -5 \\ 2 & -1 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}.$$

Now,

$$|A| = 1(12-5) + 1(9+10) + 2(-3-8) = 7 + 19 - 22 = 4 \neq 0$$

Thus, A is non-singular. Therefore, its inverse exists.

$$\text{Now, } A_{11} = 7, A_{12} = -19, A_{13} = -11$$
$$A_{21} = 1, A_{22} = -1, A_{23} = -1$$
$$A_{31} = -3, A_{32} = 11, A_{33} = 7$$

$$\therefore\ A^{-1} = \frac{1}{|A|}(adjA) = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}$$

$$\therefore X = A^{-1}B = \frac{1}{4}\begin{bmatrix} 7 & 1 & -3 \\ -19 & -1 & 11 \\ -11 & -1 & 7 \end{bmatrix}\begin{bmatrix} 7 \\ -5 \\ 12 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 49-5-36 \\ -133+5+132 \\ -77+5+84 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 8 \\ 4 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

Hence, $x = 2, y = 1, \text{and } z = 3.$

Question 15:

If $A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$, find A^{-1}. Using A^{-1} solve the system of equations

$$2x-3y+5z=11$$
$$3x+2y-4z=-5$$
$$x+y-2z=-3$$

Answer

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}$$

$$\therefore |A|=2(-4+4)+3(-6+4)+5(3-2)=0-6+5=-1\neq 0$$

$$\text{Now, } A_{11}=0,\ A_{12}=2,\ A_{13}=1$$
$$A_{21}=-1,\ A_{22}=-9,\ A_{23}=-5$$
$$A_{31}=2,\ A_{32}=23,\ A_{33}=13$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)=-\begin{bmatrix}0 & -1 & 2\\ 2 & -9 & 23\\ 1 & -5 & 13\end{bmatrix}=\begin{bmatrix}0 & 1 & -2\\ -2 & 9 & -23\\ -1 & 5 & -13\end{bmatrix} \quad ...(1)$$

Now, the given system of equations can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & -3 & 5\\ 3 & 2 & -4\\ 1 & 1 & -2\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \text{and } B=\begin{bmatrix}11\\ -5\\ -3\end{bmatrix}.$$

The solution of the system of equations is given by $X = A^{-1}B$.

$X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \qquad \left[\text{Using } (1)\right]$$

$$= \begin{bmatrix} 0-5+6 \\ -22-45+69 \\ -11-25+39 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Hence, $x = 1,\ y = 2,$ and $z = 3$.

Question 16:

The cost of 4 kg onion, 3 kg wheat and 2 kg rice is Rs 60. The cost of 2 kg onion, 4 kg wheat and 6 kg rice is Rs 90. The cost of 6 kg onion 2 kg wheat and 3 kg rice is Rs 70. Find cost of each item per kg by matrix method.

Answer

Let the cost of onions, wheat, and rice per kg be Rs *x*, Rs *y*,and Rs *z* respectively.

Then, the given situation can be represented by a system of equations as:

$$4x + 3y + 2z = 60$$
$$2x + 4y + 6z = 90$$
$$6x + 2y + 3z = 70$$

This system of equations can be written in the form of $AX = B$, where

$$A = \begin{bmatrix} 4 & 3 & 2 \\ 2 & 4 & 6 \\ 6 & 2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \text{ and } B = \begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}.$$

$$|A| = 4(12-12) - 3(6-36) + 2(4-24) = 0 + 90 - 40 = 50 \neq 0$$

Now, $A_{11} = 0, A_{12} = 30, A_{13} = -20$

$A_{21} = -5, A_{22} = 0, A_{23} = 10$

$A_{31} = 10, A_{32} = -20, A_{33} = 10$

$$\therefore adjA = \begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}$$

Now,

$X = A^{-1} B$

$$\Rightarrow X = \frac{1}{50}\begin{bmatrix} 0 & -5 & 10 \\ 30 & 0 & -20 \\ -20 & 10 & 10 \end{bmatrix}\begin{bmatrix} 60 \\ 90 \\ 70 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{50}\begin{bmatrix} 0-450+700 \\ 1800+0-1400 \\ -1200+900+700 \end{bmatrix}$$

$$= \frac{1}{50}\begin{bmatrix} 250 \\ 400 \\ 400 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 8 \\ 8 \end{bmatrix}$$

$\therefore x = 5, y = 8, \text{and } z = 8.$

Hence, the cost of onions is Rs 5 per kg, the cost of wheat is Rs 8 per kg, and the cost of rice is Rs 8 per kg.

Miscellaneous Solutions

Question 1:

Prove that the determinant $\begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$ is independent of θ.

Answer

$$\Delta = \begin{vmatrix} x & \sin\theta & \cos\theta \\ -\sin\theta & -x & 1 \\ \cos\theta & 1 & x \end{vmatrix}$$

$$= x\left(x^2 - 1\right) - \sin\theta\left(-x\sin\theta - \cos\theta\right) + \cos\theta\left(-\sin\theta + x\cos\theta\right)$$

$$= x^3 - x + x\sin^2\theta + \sin\theta\cos\theta - \sin\theta\cos\theta + x\cos^2\theta$$

$$= x^3 - x + x\left(\sin^2\theta + \cos^2\theta\right)$$

$$= x^3 - x + x$$

$$= x^3 \text{ (Independent of } \theta\text{)}$$

Hence, Δ is independent of θ.

Question 2:

Without expanding the determinant, prove that

$$\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix} = \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

Answer

$$\text{L.H.S.} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$$

$$= \frac{1}{abc}\begin{vmatrix} a^2 & a^3 & abc \\ b^2 & b^3 & abc \\ c^2 & c^3 & abc \end{vmatrix} \qquad [R_1 \to aR_1, R_2 \to bR_2, \text{and } R_3 \to cR_3]$$

$$= \frac{1}{abc}\cdot abc\begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix} \qquad [\text{Taking out factor } abc \text{ from } C_3]$$

$$= \begin{vmatrix} a^2 & a^3 & 1 \\ b^2 & b^3 & 1 \\ c^2 & c^3 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix} \qquad [\text{Applying } C_1 \leftrightarrow C_3 \text{ and } C_2 \leftrightarrow C_3]$$

$$= \text{R.H.S.}$$

Hence, the given result is proved.

Question 3:

Evaluate $\begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta & -\sin\alpha \\ -\sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta & \cos\alpha \end{vmatrix}$$

Expanding along C_3, we have:

$$\Delta = -\sin\alpha\left(-\sin\alpha\sin^2\beta - \cos^2\beta\sin\alpha\right) + \cos\alpha\left(\cos\alpha\cos^2\beta + \cos\alpha\sin^2\beta\right)$$
$$= \sin^2\alpha\left(\sin^2\beta + \cos^2\beta\right) + \cos^2\alpha\left(\cos^2\beta + \sin^2\beta\right)$$
$$= \sin^2\alpha(1) + \cos^2\alpha(1)$$
$$= 1$$

Question 4:

If a, b and c are real numbers, and $\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix} = 0$,

Show that either $a + b + c = 0$ or $a = b = c$.

Answer

$$\Delta = \begin{vmatrix} b+c & c+a & a+b \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(a+b+c) & 2(a+b+c) & 2(a+b+c) \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$
$$= 2(a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ c+a & a+b & b+c \\ a+b & b+c & c+a \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = 2(a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ c+a & b-c & b-a \\ a+b & c-a & c-b \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = 2(a+b+c)(1)\left[(b-c)(c-b)-(b-a)(c-a)\right]$$
$$= 2(a+b+c)\left[-b^2-c^2+2bc-bc+ba+ac-a^2\right]$$
$$= 2(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]$$

It is given that $\Delta = 0$.

$$(a+b+c)\left[ab+bc+ca-a^2-b^2-c^2\right]=0$$

$\Rightarrow$ Either $a+b+c=0$, or $ab+bc+ca-a^2-b^2-c^2=0$.

Now,

$$ab+bc+ca-a^2-b^2-c^2=0$$
$$\Rightarrow -2ab-2bc-2ca+2a^2+2b^2+2c^2=0$$
$$\Rightarrow (a-b)^2+(b-c)^2+(c-a)^2=0$$
$$\Rightarrow (a-b)^2=(b-c)^2=(c-a)^2=0 \qquad \left[(a-b)^2,(b-c)^2,(c-a)^2 \text{ are non-negative}\right]$$
$$\Rightarrow (a-b)=(b-c)=(c-a)=0$$
$$\Rightarrow a=b=c$$

Hence, if $\Delta = 0$, then either $a + b + c = 0$ or $a = b = c$.

Question 5:

Solve the equations $\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0, a \neq 0$

Answer

$$\begin{vmatrix} x+a & x & x \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $R_1 \to R_1 + R_2 + R_3$, we get:

$$\begin{vmatrix} 3x+a & 3x+a & 3x+a \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

$$\Rightarrow (3x+a)\begin{vmatrix} 1 & 1 & 1 \\ x & x+a & x \\ x & x & x+a \end{vmatrix} = 0$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$(3x+a)\begin{vmatrix} 1 & 0 & 0 \\ x & a & 0 \\ x & 0 & a \end{vmatrix} = 0$$

Expanding along R_1, we have:

$(3x+a)\left[1 \times a^2\right] = 0$

$\Rightarrow a^2(3x+a) = 0$

But $a \neq 0$.

Therefore, we have:

$3x + a = 0$

$\Rightarrow x = -\dfrac{a}{3}$

Question 6:

Prove that $\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$

Answer

$$\Delta=\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix}$$

Taking out common factors a, b, and c from C_1, C_2, and C_3, we have:

$$\Delta=abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=abc\begin{vmatrix} a & c & a+c \\ b & b-c & -c \\ b-a & b & -a \end{vmatrix}$$

Applying $R_2 \to R_2 + R_1$, we have:

$$\Delta=abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b-a & b & -a \end{vmatrix}$$

Applying $R_3 \to R_3 + R_2$, we have:

$$\Delta=abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 2b & 2b & 0 \end{vmatrix}$$

$$=2ab^2c\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ 1 & 1 & 0 \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$, we have:

$$\Delta=2ab^2c\begin{vmatrix} a & c-a & a+c \\ a+b & -a & a \\ 1 & 0 & 0 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta=2ab^2c\left[a(c-a)+a(a+c)\right]$$

$$=2ab^2c\left[ac-a^2+a^2+ac\right]$$

$$=2ab^2c(2ac)$$

$$=4a^2b^2c^2$$

Hence, the given result is proved.

Question 8:

Let $A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$ verify that

(i) $\left[adjA\right]^{-1}=adj\left(A^{-1}\right)$

(ii) $\left(A^{-1}\right)^{-1}=A$

Answer

$$A=\begin{bmatrix}1 & -2 & 1\\ -2 & 3 & 1\\ 1 & 1 & 5\end{bmatrix}$$

$$\therefore |A|=1(15-1)+2(-10-1)+1(-2-3)=14-22-5=-13$$

$$\begin{aligned}\text{Now, } & A_{11}=14, A_{12}=11, A_{13}=-5\\ & A_{21}=11, A_{22}=4, A_{23}=-3\\ & A_{31}=-5, A_{32}=-3, A_{13}=-1\end{aligned}$$

$$\therefore adjA=\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}$$

$$\therefore\ A^{-1}=\frac{1}{|A|}(adjA)$$

$$=-\frac{1}{13}\begin{bmatrix}14 & 11 & -5\\ 11 & 4 & -3\\ -5 & -3 & -1\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}$$

(i)

$$\begin{aligned}|adjA| &= 14(-4-9)-11(-11-15)-5(-33+20)\\ &= 14(-13)-11(-26)-5(-13)\\ &= -182+286+65=169\end{aligned}$$

We have,

$$adj\left(adjA\right)=\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$\therefore\left[adjA\right]^{-1}=\frac{1}{\left|adjA\right|}\left(adj\left(adjA\right)\right)$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}$$

$$=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

$$\text{Now, } A^{-1}=\frac{1}{13}\begin{bmatrix}-14 & -11 & 5\\ -11 & -4 & 3\\ 5 & 3 & 1\end{bmatrix}=\begin{bmatrix}-\frac{14}{13} & -\frac{11}{13} & \frac{5}{13}\\ -\frac{11}{13} & -\frac{4}{13} & \frac{3}{13}\\ \frac{5}{13} & \frac{3}{13} & \frac{1}{13}\end{bmatrix}$$

$$\therefore adj\left(A^{-1}\right)=\begin{bmatrix}-\frac{4}{169}-\frac{9}{169} & -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{33}{169}+\frac{20}{169}\\ -\left(-\frac{11}{169}-\frac{15}{169}\right) & -\frac{14}{169}-\frac{25}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right)\\ -\frac{33}{169}+\frac{20}{169} & -\left(-\frac{42}{169}+\frac{55}{169}\right) & \frac{56}{169}-\frac{121}{169}\end{bmatrix}$$

$$=\frac{1}{169}\begin{bmatrix}-13 & 26 & -13\\ 26 & -39 & -13\\ -13 & -13 & -65\end{bmatrix}=\frac{1}{13}\begin{bmatrix}-1 & 2 & -1\\ 2 & -3 & -1\\ -1 & -1 & -5\end{bmatrix}$$

Hence, $\left[adjA\right]^{-1}=adj\left(A^{-1}\right)$.

(ii)

We have shown that:

$$A^{-1} = \frac{1}{13}\begin{bmatrix} -14 & -11 & 5 \\ -11 & -4 & 3 \\ 5 & 3 & 1 \end{bmatrix}$$

And, $adjA^{-1} = \frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix}$

Now,

$$\left|A^{-1}\right| = \left(\frac{1}{13}\right)^3 \left[-14\times(-13)+11\times(-26)+5\times(-13)\right] = \left(\frac{1}{13}\right)^3 \times(-169) = -\frac{1}{13}$$

$$\therefore \left(A^{-1}\right)^{-1} = \frac{adjA^{-1}}{\left|A^{-1}\right|} = \frac{1}{\left(-\frac{1}{13}\right)}\times\frac{1}{13}\begin{bmatrix} -1 & 2 & -1 \\ 2 & -3 & -1 \\ -1 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 3 & 1 \\ 1 & 1 & 5 \end{bmatrix} = A$$

$$\therefore \left(A^{-1}\right)^{-1} = A$$

Question 9:

Evaluate $\begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} x & y & x+y \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $R_1 \to R_1 + R_2 + R_3$, we have:

$$\Delta = \begin{vmatrix} 2(x+y) & 2(x+y) & 2(x+y) \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

$$= 2(x+y)\begin{vmatrix} 1 & 1 & 1 \\ y & x+y & x \\ x+y & x & y \end{vmatrix}$$

Applying $C_2 \to C_2 - C_1$ and $C_3 \to C_3 - C_1$, we have:

$$\Delta = 2(x+y)\begin{vmatrix} 1 & 0 & 0 \\ y & x & x-y \\ x+y & -y & -x \end{vmatrix}$$

Expanding along R_1, we have:

$$\Delta = 2(x+y)\left[-x^2 + y(x-y)\right]$$

$$= -2(x+y)\left(x^2 + y^2 - yx\right)$$

$$= -2\left(x^3 + y^3\right)$$

Question 10:

Evaluate $\begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$

Answer

$$\Delta = \begin{vmatrix} 1 & x & y \\ 1 & x+y & y \\ 1 & x & x+y \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} 1 & x & y \\ 0 & y & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = 1(xy - 0) = xy$$

Question 11:

Using properties of determinants, prove that:

$$\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix} = (\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)(\alpha+\beta+\gamma)$$

Answer

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta & \beta^2 & \gamma+\alpha \\ \gamma & \gamma^2 & \alpha+\beta \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ \beta-\alpha & \beta^2-\alpha^2 & \alpha-\beta \\ \gamma-\alpha & \gamma^2-\alpha^2 & \alpha-\gamma \end{vmatrix}$$

$$= (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 1 & \gamma+\alpha & -1 \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we have:

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\begin{vmatrix} \alpha & \alpha^2 & \beta+\gamma \\ 1 & \beta+\alpha & -1 \\ 0 & \gamma-\beta & 0 \end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta = (\beta-\alpha)(\gamma-\alpha)\left[-(\gamma-\beta)(-\alpha-\beta-\gamma)\right]$$
$$= (\beta-\alpha)(\gamma-\alpha)(\gamma-\beta)(\alpha+\beta+\gamma)$$
$$= (\alpha-\beta)(\beta-\gamma)(\gamma-\alpha)(\alpha+\beta+\gamma)$$

Hence, the given result is proved.

Question 12:

Using properties of determinants, prove that:

$$\begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix} = (1+pxyz)(x-y)(y-z)(z-x)$$

Answer

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y & y^2 & 1+py^3 \\ z & z^2 & 1+pz^3 \end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta = \begin{vmatrix} x & x^2 & 1+px^3 \\ y-x & y^2-x^2 & p\left(y^3-x^3\right) \\ z-x & z^2-x^2 & p\left(z^3-x^3\right) \end{vmatrix}$$

$$= (y-x)(z-x)\begin{vmatrix} x & x^2 & 1+px^3 \\ 1 & y+x & p\left(y^2+x^2+xy\right) \\ 1 & z+x & p\left(z^2+x^2+xz\right) \end{vmatrix}$$

Applying $R_3 \to R_3 - R_2$, we have:

$$\Delta=(y-x)(z-x)\begin{vmatrix}x & x^2 & 1+px^3\\ 1 & y+x & p(y^2+x^2+xy)\\ 0 & z-y & p(z-y)(x+y+z)\end{vmatrix}$$

$$=(y-x)(z-x)(z-y)\begin{vmatrix}x & x^2 & 1+px^3\\ 1 & y+x & p(y^2+x^2+xy)\\ 0 & 1 & p(x+y+z)\end{vmatrix}$$

Expanding along R_3, we have:

$$\Delta=(x-y)(y-z)(z-x)\left[(-1)(p)(xy^2+x^3+x^2y)+1+px^3+p(x+y+z)(xy)\right]$$

$$=(x-y)(y-z)(z-x)\left[-pxy^2-px^3-px^2y+1+px^3+px^2y+pxy^2+pxyz\right]$$

$$=(x-y)(y-z)(z-x)(1+pxyz)$$

Hence, the given result is proved.

Question 13:

Using properties of determinants, prove that:

$$\begin{vmatrix}3a & -a+b & -a+c\\ -b+a & 3b & -b+c\\ -c+a & -c+b & 3c\end{vmatrix}=3(a+b+c)(ab+bc+ca)$$

Answer

$$\Delta=\begin{vmatrix}3a & -a+b & -a+c\\ -b+a & 3b & -b+c\\ -c+a & -c+b & 3c\end{vmatrix}$$

Applying $C_1 \to C_1+C_2+C_3$, we have:

$$\Delta=\begin{vmatrix}a+b+c & -a+b & -a+c\\ a+b+c & 3b & -b+c\\ a+b+c & -c+b & 3c\end{vmatrix}$$

$$=(a+b+c)\begin{vmatrix}1 & -a+b & -a+c\\ 1 & 3b & -b+c\\ 1 & -c+b & 3c\end{vmatrix}$$

Applying $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$, we have:

$$\Delta=(a+b+c)\begin{vmatrix}1 & -a+b & -a+c\\ 0 & 2b+a & a-b\\ 0 & a-c & 2c+a\end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta=(a+b+c)\left[(2b+a)(2c+a)-(a-b)(a-c)\right]$$

$$=(a+b+c)\left[4bc+2ab+2ac+a^2-a^2+ac+ba-bc\right]$$

$$=(a+b+c)(3ab+3bc+3ac)$$

$$=3(a+b+c)(ab+bc+ca)$$

Hence, the given result is proved.

Question 14:

Using properties of determinants, prove that:

$$\begin{vmatrix}1 & 1+p & 1+p+q\\ 2 & 3+2p & 4+3p+2q\\ 3 & 6+3p & 10+6p+3q\end{vmatrix}=1$$

Answer

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 2 & 3+2p & 4+3p+2q \\ 3 & 6+3p & 10+6p+3q \end{vmatrix}$$

Applying $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - 3R_1$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 3 & 7+3p \end{vmatrix}$$

Applying $R_3 \to R_3 - 3R_2$, we have:

$$\Delta = \begin{vmatrix} 1 & 1+p & 1+p+q \\ 0 & 1 & 2+p \\ 0 & 0 & 1 \end{vmatrix}$$

Expanding along C_1, we have:

$$\Delta = 1\begin{vmatrix} 1 & 2+p \\ 0 & 1 \end{vmatrix} = 1(1-0) = 1$$

Hence, the given result is proved.

Question 15:

Using properties of determinants, prove that:

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix} = 0$$

Answer

$$\Delta = \begin{vmatrix} \sin\alpha & \cos\alpha & \cos(\alpha+\delta) \\ \sin\beta & \cos\beta & \cos(\beta+\delta) \\ \sin\gamma & \cos\gamma & \cos(\gamma+\delta) \end{vmatrix}$$

$$= \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \sin\alpha\sin\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \sin\beta\sin\delta & \cos\beta\cos\delta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \sin\gamma\sin\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Applying $C_1 \to C_1 + C_3$, we have:

$$\Delta = \frac{1}{\sin\delta\cos\delta}\begin{vmatrix} \cos\alpha\cos\delta & \cos\alpha\cos\delta & \cos\alpha\cos\delta - \sin\alpha\sin\delta \\ \cos\beta\cos\delta & \cos\beta\cos\delta & \cos\beta\cos\delta - \sin\beta\sin\delta \\ \cos\gamma\cos\delta & \cos\gamma\cos\delta & \cos\gamma\cos\delta - \sin\gamma\sin\delta \end{vmatrix}$$

Here, two columns C_1 and C_2 are identical.

$\therefore \Delta = 0.$

Hence, the given result is proved.

Question 16:

Solve the system of the following equations

$$\frac{2}{x}+\frac{3}{y}+\frac{10}{z}=4$$

$$\frac{4}{x}-\frac{6}{y}+\frac{5}{z}=1$$

$$\frac{6}{x}+\frac{9}{y}-\frac{20}{z}=2$$

Answer

Let $\frac{1}{x}=p, \frac{1}{y}=q, \frac{1}{z}=r.$

Then the given system of equations is as follows:

$$2p+3q+10r=4$$

$$4p-6q+5r=1$$

$$6p+9q-20r=2$$

This system can be written in the form of $AX = B$, where

$$A=\begin{bmatrix}2 & 3 & 10\\ 4 & -6 & 5\\ 6 & 9 & -20\end{bmatrix}, X=\begin{bmatrix}p\\ q\\ r\end{bmatrix} \text{ and } B=\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}.$$

Now,

$$\begin{aligned}|A| &= 2(120-45)-3(-80-30)+10(36+36)\\ &=150+330+720\\ &=1200\end{aligned}$$

A

Thus, A is non-singular. Therefore, its inverse exists.

Now,

A_{11} = 75, A_{12} = 110, A_{13} = 72

A_{21} = 150, A_{22} = −100, A_{23} = 0

A_{31} = 75, A_{32} = 30, A_{33} = − 24

$$\therefore A^{-1}=\frac{1}{|A|}adjA$$

$$=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}$$

Now,

$$X=A^{-1}B$$

$$\Rightarrow\begin{bmatrix}p\\ q\\ r\end{bmatrix}=\frac{1}{1200}\begin{bmatrix}75 & 150 & 75\\ 110 & -100 & 30\\ 72 & 0 & -24\end{bmatrix}\begin{bmatrix}4\\ 1\\ 2\end{bmatrix}$$

$$=\frac{1}{1200}\begin{bmatrix}300+150+150\\ 440-100+60\\ 288+0-48\end{bmatrix}$$

$$=\frac{1}{1200}\begin{bmatrix}600\\ 400\\ 240\end{bmatrix}=\begin{bmatrix}\frac{1}{2}\\ \frac{1}{3}\\ \frac{1}{5}\end{bmatrix}$$

$$\therefore p=\frac{1}{2}, q=\frac{1}{3}, \text{ and } r=\frac{1}{5}$$

Hence, $x=2, y=3,$ and $z=5$.

Question 17:

Choose the correct answer.

If *a*, *b*, *c*, are in A.P., then the determinant

$$\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

A. 0 **B.** 1 **C.** *x* **D.** 2*x*

Answer

Answer: A

$$\Delta = \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$$

$$= \begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+(a+c) \\ x+4 & x+5 & x+2c \end{vmatrix} \qquad (2b = a+c \text{ as } a, b, \text{and } c \text{ are in A.P.})$$

Applying $R_1 \to R_1 - R_2$ and $R_3 \to R_3 - R_2$, we have:

$$\Delta = \begin{vmatrix} -1 & -1 & a-c \\ x+3 & x+4 & x+(a+c) \\ 1 & 1 & c-a \end{vmatrix}$$

Applying $R_1 \to R_1 + R_3$, we have:

$$\Delta = \begin{vmatrix} 0 & 0 & 0 \\ x+3 & x+4 & x+a+c \\ 1 & 1 & c-a \end{vmatrix}$$

Here, all the elements of the first row (R_1) are zero.

Hence, we have $\Delta = 0$.

The correct answer is A.

Question 18:

Choose the correct answer.

If x, y, z are nonzero real numbers, then the inverse of matrix $A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ is

A. $\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$ **B.** $xyz\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$

C. $\dfrac{1}{xyz}\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$ **D.** $\dfrac{1}{xyz}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Answer

Answer: A

$$A=\begin{bmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{bmatrix}$$

$$\therefore |A| = x(yz-0) = xyz \neq 0$$

$$\begin{aligned} \text{Now, } & A_{11} = yz, A_{12} = 0, A_{13} = 0 \\ & A_{21} = 0, A_{22} = xz, A_{23} = 0 \\ & A_{31} = 0, A_{32} = 0, A_{33} = xy \end{aligned}$$

$$\therefore adjA = \begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{|A|} adjA$$

$$=\frac{1}{xyz}\begin{bmatrix} yz & 0 & 0 \\ 0 & xz & 0 \\ 0 & 0 & xy \end{bmatrix}$$

$$=\begin{bmatrix} \frac{yz}{xyz} & 0 & 0 \\ 0 & \frac{xz}{xyz} & 0 \\ 0 & 0 & \frac{xy}{xyz} \end{bmatrix}$$

$$=\begin{bmatrix} \frac{1}{x} & 0 & 0 \\ 0 & \frac{1}{y} & 0 \\ 0 & 0 & \frac{1}{z} \end{bmatrix}=\begin{bmatrix} x^{-1} & 0 & 0 \\ 0 & y^{-1} & 0 \\ 0 & 0 & z^{-1} \end{bmatrix}$$

The correct answer is A.

Question 19:

Choose the correct answer.

Let $A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$, where $0 \le \theta \le 2\pi$, then

A. Det (A) = 0

B. Det (A) ∈ $(2, \infty)$

C. Det (A) ∈ (2, 4)

D. Det (A)∈ [2, 4]

Answer

sAnswer: D

$$A=\begin{bmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{bmatrix}$$

$$\therefore |A| = 1\left(1+\sin^2\theta\right)-\sin\theta\left(-\sin\theta+\sin\theta\right)+1\left(\sin^2\theta+1\right)$$

$$= 1+\sin^2\theta+\sin^2\theta+1$$

$$= 2+2\sin^2\theta$$

$$= 2\left(1+\sin^2\theta\right)$$

$$\text{Now, } 0 \le \theta \le 2\pi$$

$$\Rightarrow 0 \le \sin\theta \le 1$$

$$\Rightarrow 0 \le \sin^2\theta \le 1$$

$$\Rightarrow 1 \le 1+\sin^2\theta \le 2$$

$$\Rightarrow 2 \le 2\left(1+\sin^2\theta\right) \le 4$$

$$\therefore \text{Det}(A) \in [2,4]$$

The correct answer is D.