

## Exercise 11.1

**Question 1:**

If a line makes angles  $90^\circ$ ,  $135^\circ$ ,  $45^\circ$  with  $x$ ,  $y$  and  $z$ -axes respectively, find its direction cosines.

Answer

Let direction cosines of the line be  $l$ ,  $m$ , and  $n$ .

$$l = \cos 90^\circ = 0$$

$$m = \cos 135^\circ = -\frac{1}{\sqrt{2}}$$

$$n = \cos 45^\circ = \frac{1}{\sqrt{2}}$$

Therefore, the direction cosines of the line are  $0, -\frac{1}{\sqrt{2}}$ , and  $\frac{1}{\sqrt{2}}$ .

**Question 2:**

Find the direction cosines of a line which makes equal angles with the coordinate axes.

Answer

Let the direction cosines of the line make an angle  $a$  with each of the coordinate axes.

$$\therefore l = \cos a, m = \cos a, n = \cos a$$

$$l^2 + m^2 + n^2 = 1$$

$$\Rightarrow \cos^2 \alpha + \cos^2 \alpha + \cos^2 \alpha = 1$$

$$\Rightarrow 3 \cos^2 \alpha = 1$$

$$\Rightarrow \cos^2 \alpha = \frac{1}{3}$$

$$\Rightarrow \cos \alpha = \pm \frac{1}{\sqrt{3}}$$

Thus, the direction cosines of the line, which is equally inclined to the coordinate axes,

are  $\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}},$  and  $\pm \frac{1}{\sqrt{3}}.$

### Question 3:

If a line has the direction ratios  $-18, 12, -4,$  then what are its direction cosines?

Answer

If a line has direction ratios of  $-18, 12,$  and  $-4,$  then its direction cosines are

$$\frac{-18}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{12}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-18)^2 + (12)^2 + (-4)^2}}$$

i.e.,  $\frac{-18}{22}, \frac{12}{22}, \frac{-4}{22}$

$$\frac{-9}{11}, \frac{6}{11}, \frac{-2}{11}$$

Thus, the direction cosines are  $-\frac{9}{11}, \frac{6}{11},$  and  $\frac{-2}{11}.$

### Question 4:

Show that the points  $(2, 3, 4), (-1, -2, 1), (5, 8, 7)$  are collinear.

Answer

The given points are A  $(2, 3, 4),$  B  $(-1, -2, 1),$  and C  $(5, 8, 7).$

It is known that the direction ratios of line joining the points,  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2),$  are given by,  $x_2 - x_1, y_2 - y_1,$  and  $z_2 - z_1.$

The direction ratios of AB are  $(-1 - 2), (-2 - 3),$  and  $(1 - 4)$  i.e.,  $-3, -5,$  and  $-3.$

The direction ratios of BC are  $(5 - (-1)), (8 - (-2)),$  and  $(7 - 1)$  i.e.,  $6, 10,$  and  $6.$

It can be seen that the direction ratios of BC are  $-2$  times that of AB i.e., they are proportional.

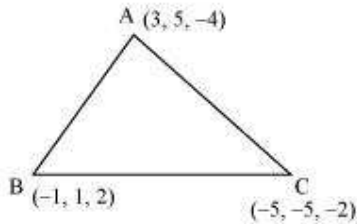
Therefore, AB is parallel to BC. Since point B is common to both AB and BC, points A, B, and C are collinear.

**Question 5:**

Find the direction cosines of the sides of the triangle whose vertices are  $(3, 5, -4)$ ,  $(-1, 1, 2)$  and  $(-5, -5, -2)$

Answer

The vertices of  $\triangle ABC$  are A  $(3, 5, -4)$ , B  $(-1, 1, 2)$ , and C  $(-5, -5, -2)$ .



The direction ratios of side AB are  $(-1 - 3)$ ,  $(1 - 5)$ , and  $(2 - (-4))$  i.e.,  $-4$ ,  $-4$ , and  $6$ .

$$\begin{aligned}\text{Then, } \sqrt{(-4)^2 + (-4)^2 + (6)^2} &= \sqrt{16 + 16 + 36} \\ &= \sqrt{68} \\ &= 2\sqrt{17}\end{aligned}$$

Therefore, the direction cosines of AB are

$$\begin{aligned}\frac{-4}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}, \frac{-4}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}}, \frac{6}{\sqrt{(-4)^2 + (-4)^2 + (6)^2}} \\ \frac{-4}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}, \frac{6}{2\sqrt{17}} \\ \frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{3}{\sqrt{17}}\end{aligned}$$

The direction ratios of BC are  $(-5 - (-1))$ ,  $(-5 - 1)$ , and  $(-2 - 2)$  i.e.,  $-4$ ,  $-6$ , and  $-4$ .

Therefore, the direction cosines of BC are

$$\begin{aligned}\frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}, \frac{-6}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}}, \frac{-4}{\sqrt{(-4)^2 + (-6)^2 + (-4)^2}} \\ \frac{-4}{2\sqrt{17}}, \frac{-6}{2\sqrt{17}}, \frac{-4}{2\sqrt{17}}\end{aligned}$$

The direction ratios of CA are  $(-5 - 3)$ ,  $(-5 - 5)$ , and  $(-2 - (-4))$  i.e.,  $-8$ ,  $-10$ , and  $2$ .

Therefore, the direction cosines of AC are

$$\frac{-8}{\sqrt{(-8)^2 + (10)^2 + (2)^2}}, \frac{-5}{\sqrt{(-8)^2 + (10)^2 + (2)^2}}, \frac{2}{\sqrt{(-8)^2 + (10)^2 + (2)^2}}$$
$$\frac{-8}{2\sqrt{42}}, \frac{-10}{2\sqrt{42}}, \frac{2}{2\sqrt{42}}$$

## Exercise 11.2

**Question 1:**

Show that the three lines with direction cosines

$\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}; \frac{4}{13}, \frac{12}{13}, \frac{3}{13}; \frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$  are mutually perpendicular.

Answer

Two lines with direction cosines,  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$ , are perpendicular to each other, if  $l_1l_2 + m_1m_2 + n_1n_2 = 0$

(i) For the lines with direction cosines,  $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$  and  $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$ , we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{12}{13} \times \frac{4}{13} + \left(\frac{-3}{13}\right) \times \frac{12}{13} + \left(\frac{-4}{13}\right) \times \frac{3}{13} \\ &= \frac{48}{169} - \frac{36}{169} - \frac{12}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

(ii) For the lines with direction cosines,  $\frac{4}{13}, \frac{12}{13}, \frac{3}{13}$  and  $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$ , we obtain

$$\begin{aligned} l_1l_2 + m_1m_2 + n_1n_2 &= \frac{4}{13} \times \frac{3}{13} + \frac{12}{13} \times \left(\frac{-4}{13}\right) + \frac{3}{13} \times \frac{12}{13} \\ &= \frac{12}{169} - \frac{48}{169} + \frac{36}{169} \\ &= 0 \end{aligned}$$

Therefore, the lines are perpendicular.

(iii) For the lines with direction cosines,  $\frac{3}{13}, \frac{-4}{13}, \frac{12}{13}$  and  $\frac{12}{13}, \frac{-3}{13}, \frac{-4}{13}$ , we obtain

$$\begin{aligned}
 l_1l_2 + m_1m_2 + n_1n_2 &= \left(\frac{3}{13}\right) \times \left(\frac{12}{13}\right) + \left(\frac{-4}{13}\right) \times \left(\frac{-3}{13}\right) + \left(\frac{12}{13}\right) \times \left(\frac{-4}{13}\right) \\
 &= \frac{36}{169} + \frac{12}{169} - \frac{48}{169} \\
 &= 0
 \end{aligned}$$

Therefore, the lines are perpendicular.

Thus, all the lines are mutually perpendicular.

### Question 2:

Show that the line through the points  $(1, -1, 2)$   $(3, 4, -2)$  is perpendicular to the line through the points  $(0, 3, 2)$  and  $(3, 5, 6)$ .

Answer

Let AB be the line joining the points,  $(1, -1, 2)$  and  $(3, 4, -2)$ , and CD be the line joining the points,  $(0, 3, 2)$  and  $(3, 5, 6)$ .

The direction ratios,  $a_1, b_1, c_1$ , of AB are  $(3 - 1)$ ,  $(4 - (-1))$ , and  $(-2 - 2)$  i.e., 2, 5, and -4.

The direction ratios,  $a_2, b_2, c_2$ , of CD are  $(3 - 0)$ ,  $(5 - 3)$ , and  $(6 - 2)$  i.e., 3, 2, and 4.

AB and CD will be perpendicular to each other, if  $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + 5 \times 2 + (-4) \times 4$$

$$= 6 + 10 - 16$$

$$= 0$$

Therefore, AB and CD are perpendicular to each other.

### Question 3:

Show that the line through the points  $(4, 7, 8)$   $(2, 3, 4)$  is parallel to the line through the points  $(-1, -2, 1)$ ,  $(1, 2, 5)$ .

Answer

Let AB be the line through the points,  $(4, 7, 8)$  and  $(2, 3, 4)$ , and CD be the line through the points,  $(-1, -2, 1)$  and  $(1, 2, 5)$ .

The directions ratios,  $a_1, b_1, c_1$ , of AB are  $(2 - 4)$ ,  $(3 - 7)$ , and  $(4 - 8)$  i.e., -2, -4, and -4.

The direction ratios,  $a_2, b_2, c_2$ , of CD are  $(1 - (-1))$ ,  $(2 - (-2))$ , and  $(5 - 1)$  i.e., 2, 4, and 4.

AB will be parallel to CD, if  $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

$$\frac{a_1}{a_2} = \frac{-2}{2} = -1$$

$$\frac{b_1}{b_2} = \frac{-4}{4} = -1$$

$$\frac{c_1}{c_2} = \frac{-4}{4} = -1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, AB is parallel to CD.

#### Question 4:

Find the equation of the line which passes through the point (1, 2, 3) and is parallel to the vector  $3\hat{i} + 2\hat{j} - 2\hat{k}$ .

Answer

It is given that the line passes through the point A (1, 2, 3). Therefore, the position vector through A is  $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$$\vec{b} = 3\hat{i} + 2\hat{j} - 2\hat{k}$$

It is known that the line which passes through point A and parallel to  $\vec{b}$  is given by

$\vec{r} = \vec{a} + \lambda\vec{b}$ , where  $\lambda$  is a constant.

$$\Rightarrow \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(3\hat{i} + 2\hat{j} - 2\hat{k})$$

This is the required equation of the line.

#### Question 5:

Find the equation of the line in vector and in Cartesian form that passes through the point with position vector  $2\hat{i} - \hat{j} + 4\hat{k}$  and is in the direction  $\hat{i} + 2\hat{j} - \hat{k}$ .

Answer

It is given that the line passes through the point with position vector

$$\vec{a} = 2\hat{i} - \hat{j} + 4\hat{k} \quad \dots(1)$$

$$\vec{b} = \hat{i} + 2\hat{j} - \hat{k} \quad \dots(2)$$

It is known that a line through a point with position vector  $\vec{a}$  and parallel to  $\vec{b}$  is given by the equation,  $\vec{r} = \vec{a} + \lambda\vec{b}$

$$\Rightarrow \vec{r} = 2\hat{i} - \hat{j} + 4\hat{k} + \lambda(\hat{i} + 2\hat{j} - \hat{k})$$

This is the required equation of the line in vector form.

$$\vec{r} = x\hat{i} - y\hat{j} + z\hat{k}$$

$$\Rightarrow x\hat{i} - y\hat{j} + z\hat{k} = (\lambda + 2)\hat{i} + (2\lambda - 1)\hat{j} + (-\lambda + 4)\hat{k}$$

Eliminating  $\lambda$ , we obtain the Cartesian form equation as

$$\frac{x-2}{1} = \frac{y+1}{2} = \frac{z-4}{-1}$$

This is the required equation of the given line in Cartesian form.

### Question 6:

Find the Cartesian equation of the line which passes through the point

$$(-2, 4, -5) \text{ and parallel to the line given by } \frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

Answer

It is given that the line passes through the point  $(-2, 4, -5)$  and is parallel to

$$\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$$

The direction ratios of the line,  $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$ , are 3, 5, and 6.

The required line is parallel to  $\frac{x+3}{3} = \frac{y-4}{5} = \frac{z+8}{6}$

Therefore, its direction ratios are  $3k$ ,  $5k$ , and  $6k$ , where  $k \neq 0$

It is known that the equation of the line through the point  $(x_1, y_1, z_1)$  and with direction

ratios,  $a, b, c$ , is given by  $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$



Therefore the equation of the required line is

$$\frac{x+2}{3k} = \frac{y-4}{5k} = \frac{z+5}{6k}$$

$$\Rightarrow \frac{x+2}{3} = \frac{y-4}{5} = \frac{z+5}{6} = k$$

**Question 7:**

The Cartesian equation of a line is  $\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2}$ . Write its vector form.

Answer

The Cartesian equation of the line is

$$\frac{x-5}{3} = \frac{y+4}{7} = \frac{z-6}{2} \quad \dots (1)$$

The given line passes through the point  $(5, -4, 6)$ . The position vector of this point is

$$\vec{a} = 5\hat{i} - 4\hat{j} + 6\hat{k}$$

Also, the direction ratios of the given line are 3, 7, and 2.

This means that the line is in the direction of vector,  $\vec{b} = 3\hat{i} + 7\hat{j} + 2\hat{k}$

It is known that the line through position vector  $\vec{a}$  and in the direction of the vector  $\vec{b}$  is

given by the equation,  $\vec{r} = \vec{a} + \lambda\vec{b}, \lambda \in R$

$$\Rightarrow \vec{r} = (5\hat{i} - 4\hat{j} + 6\hat{k}) + \lambda(3\hat{i} + 7\hat{j} + 2\hat{k})$$

This is the required equation of the given line in vector form.

**Question 8:**

Find the vector and the Cartesian equations of the lines that pass through the origin and  $(5, -2, 3)$ .

Answer

The required line passes through the origin. Therefore, its position vector is given by,

$$\vec{a} = \vec{0} \quad \dots (1)$$

The direction ratios of the line through origin and  $(5, -2, 3)$  are

$$(5 - 0) = 5, (-2 - 0) = -2, (3 - 0) = 3$$

The line is parallel to the vector given by the equation,  $\vec{b} = 5\hat{i} - 2\hat{j} + 3\hat{k}$

The equation of the line in vector form through a point with position vector  $\vec{a}$  and parallel to  $\vec{b}$  is,  $\vec{r} = \vec{a} + \lambda\vec{b}$ ,  $\lambda \in R$

$$\Rightarrow \vec{r} = \vec{0} + \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$$

$$\Rightarrow \vec{r} = \lambda(5\hat{i} - 2\hat{j} + 3\hat{k})$$

The equation of the line through the point  $(x_1, y_1, z_1)$  and direction ratios  $a, b, c$  is given

$$\text{by, } \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Therefore, the equation of the required line in the Cartesian form is

$$\frac{x-0}{5} = \frac{y-0}{-2} = \frac{z-0}{3}$$

$$\Rightarrow \frac{x}{5} = \frac{y}{-2} = \frac{z}{3}$$

### Question 9:

Find the vector and the Cartesian equations of the line that passes through the points  $(3, -2, -5)$ ,  $(3, -2, 6)$ .

Answer

Let the line passing through the points, P  $(3, -2, -5)$  and Q  $(3, -2, 6)$ , be PQ.

Since PQ passes through P  $(3, -2, -5)$ , its position vector is given by,

$$\vec{a} = 3\hat{i} - 2\hat{j} - 5\hat{k}$$

The direction ratios of PQ are given by,

$$(3 - 3) = 0, (-2 + 2) = 0, (6 + 5) = 11$$

The equation of the vector in the direction of PQ is

$$\vec{b} = 0\hat{i} - 0\hat{j} + 11\hat{k} = 11\hat{k}$$

The equation of PQ in vector form is given by,  $\vec{r} = \vec{a} + \lambda\vec{b}$ ,  $\lambda \in R$

$$\Rightarrow \vec{r} = (3\hat{i} - 2\hat{j} - 5\hat{k}) + 11\lambda\hat{k}$$

The equation of PQ in Cartesian form is

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} \quad \text{i.e.,}$$

$$\frac{x-3}{0} = \frac{y+2}{0} = \frac{z+5}{11}$$

**Question 10:**

Find the angle between the following pairs of lines:

(i)  $\vec{r} = 2\hat{i} - 5\hat{j} + \hat{k} + \lambda(3\hat{i} - 2\hat{j} + 6\hat{k})$  and

$$\vec{r} = 7\hat{i} - 6\hat{k} + \mu(\hat{i} + 2\hat{j} + 2\hat{k})$$

(ii)  $\vec{r} = 3\hat{i} + \hat{j} - 2\hat{k} + \lambda(\hat{i} - \hat{j} - 2\hat{k})$  and

$$\vec{r} = 2\hat{i} - \hat{j} - 56\hat{k} + \mu(3\hat{i} - 5\hat{j} - 4\hat{k})$$

Answer

(i) Let  $Q$  be the angle between the given lines.

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

The angle between the given pairs of lines is given by,

The given lines are parallel to the vectors,  $\vec{b}_1 = 3\hat{i} + 2\hat{j} + 6\hat{k}$  and  $\vec{b}_2 = \hat{i} + 2\hat{j} + 2\hat{k}$ , respectively.

$$\therefore |\vec{b}_1| = \sqrt{3^2 + 2^2 + 6^2} = 7$$

$$|\vec{b}_2| = \sqrt{(1)^2 + (2)^2 + (2)^2} = 3$$

$$\vec{b}_1 \cdot \vec{b}_2 = (3\hat{i} + 2\hat{j} + 6\hat{k}) \cdot (\hat{i} + 2\hat{j} + 2\hat{k})$$

$$= 3 \times 1 + 2 \times 2 + 6 \times 2$$

$$= 3 + 4 + 12$$

$$= 19$$

$$\Rightarrow \cos Q = \frac{19}{7 \times 3}$$

$$\Rightarrow Q = \cos^{-1} \left( \frac{19}{21} \right)$$

(ii) The given lines are parallel to the vectors,  $\vec{b}_1 = \hat{i} - \hat{j} - 2\hat{k}$  and  $\vec{b}_2 = 3\hat{i} - 5\hat{j} - 4\hat{k}$ , respectively.

$$\therefore |\vec{b}_1| = \sqrt{(1)^2 + (-1)^2 + (-2)^2} = \sqrt{6}$$

$$|\vec{b}_2| = \sqrt{(3)^2 + (-5)^2 + (-4)^2} = \sqrt{50} = 5\sqrt{2}$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (\hat{i} - \hat{j} - 2\hat{k}) \cdot (3\hat{i} - 5\hat{j} - 4\hat{k}) \\ &= 1 \cdot 3 - 1(-5) - 2(-4) \\ &= 3 + 5 + 8 \\ &= 16 \end{aligned}$$

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

$$\Rightarrow \cos Q = \frac{16}{\sqrt{6} \cdot 5\sqrt{2}} = \frac{16}{\sqrt{2} \cdot \sqrt{3} \cdot 5\sqrt{2}} = \frac{16}{10\sqrt{3}}$$

$$\Rightarrow \cos Q = \frac{8}{5\sqrt{3}}$$

$$\Rightarrow Q = \cos^{-1} \left( \frac{8}{5\sqrt{3}} \right)$$

### Question 11:

Find the angle between the following pairs of lines:

(i)  $\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3}$  and  $\frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}$

(ii)  $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$  and  $\frac{x-5}{4} = \frac{y-2}{1} = \frac{z-3}{8}$

i. Answer

ii. Let  $\vec{b}_1$  and  $\vec{b}_2$  be the vectors parallel to the pair of lines,

$$\frac{x-2}{2} = \frac{y-1}{5} = \frac{z+3}{-3} \text{ and } \frac{x+2}{-1} = \frac{y-4}{8} = \frac{z-5}{4}, \text{ respectively.}$$

$$\therefore \vec{b}_1 = 2\hat{i} + 5\hat{j} - 3\hat{k} \text{ and } \vec{b}_2 = -\hat{i} + 8\hat{j} + 4\hat{k}$$

$$|\vec{b}_1| = \sqrt{(2)^2 + (5)^2 + (-3)^2} = \sqrt{38}$$

$$|\vec{b}_2| = \sqrt{(-1)^2 + (8)^2 + (4)^2} = \sqrt{81} = 9$$

$$\begin{aligned} \vec{b}_1 \cdot \vec{b}_2 &= (2\hat{i} + 5\hat{j} - 3\hat{k}) \cdot (-\hat{i} + 8\hat{j} + 4\hat{k}) \\ &= 2(-1) + 5 \times 8 + (-3) \cdot 4 \\ &= -2 + 40 - 12 \\ &= 26 \end{aligned}$$

The angle,  $Q$ , between the given pair of lines is given by the relation,

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

$$\Rightarrow \cos Q = \frac{26}{9\sqrt{38}}$$

$$\Rightarrow Q = \cos^{-1} \left( \frac{26}{9\sqrt{38}} \right)$$

(ii) Let  $\vec{b}_1, \vec{b}_2$  be the vectors parallel to the given pair of lines,  $\frac{x}{2} = \frac{y}{2} = \frac{z}{1}$  and

$$\frac{x-5}{4} = \frac{y-5}{1} = \frac{z-3}{8}, \text{ respectively.}$$

$$\vec{b}_1 = 2\hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b}_2 = 4\hat{i} + \hat{j} + 8\hat{k}$$

$$\therefore |\vec{b}_1| = \sqrt{(2)^2 + (2)^2 + (1)^2} = \sqrt{9} = 3$$

$$|\vec{b}_2| = \sqrt{4^2 + 1^2 + 8^2} = \sqrt{81} = 9$$

$$\vec{b}_1 \cdot \vec{b}_2 = (2\hat{i} + 2\hat{j} + \hat{k}) \cdot (4\hat{i} + \hat{j} + 8\hat{k})$$

$$= 2 \times 4 + 2 \times 1 + 1 \times 8$$

$$= 8 + 2 + 8$$

$$= 18$$

$$\cos Q = \frac{|\vec{b}_1 \cdot \vec{b}_2|}{|\vec{b}_1| |\vec{b}_2|}$$

If  $Q$  is the angle between the given pair of lines, then

$$\Rightarrow \cos Q = \frac{18}{3 \times 9} = \frac{2}{3}$$

$$\Rightarrow Q = \cos^{-1}\left(\frac{2}{3}\right)$$

### Question 12:

$$\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{2}$$

Find the values of  $p$  so the line

$$\frac{7-7x}{3p} = \frac{y-5}{1} = \frac{6-z}{5}$$

are at right angles.

Answer

The given equations can be written in the standard form as

$$\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{2} \quad \text{and} \quad \frac{x-1}{\frac{-3p}{7}} = \frac{y-5}{1} = \frac{z-6}{-5}$$

The direction ratios of the lines are  $-3, \frac{2p}{7}, 2$  and  $\frac{-3p}{7}, 1, -5$  respectively.

Two lines with direction ratios,  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$ , are perpendicular to each other, if

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$\therefore (-3) \cdot \left(\frac{-3p}{7}\right) + \left(\frac{2p}{7}\right) \cdot (1) + 2 \cdot (-5) = 0$$

$$\Rightarrow \frac{9p}{7} + \frac{2p}{7} = 10$$

$$\Rightarrow 11p = 70$$

$$\Rightarrow p = \frac{70}{11}$$

Thus, the value of  $p$  is  $\frac{70}{11}$ .

**Question 13:**

Show that the lines  $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$  and  $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$  are perpendicular to each other.

Answer

The equations of the given lines are  $\frac{x-5}{7} = \frac{y+2}{-5} = \frac{z}{1}$  and  $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$

The direction ratios of the given lines are 7, -5, 1 and 1, 2, 3 respectively.

Two lines with direction ratios,  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$ , are perpendicular to each other, if

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$\therefore 7 \times 1 + (-5) \times 2 + 1 \times 3$$

$$= 7 - 10 + 3$$

$$= 0$$

Therefore, the given lines are perpendicular to each other.

**Question 14:**

Find the shortest distance between the lines

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k}) \text{ and}$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

Answer

The equations of the given lines are

$$\vec{r} = (\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} - \hat{j} + \hat{k})$$

$$\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$$

It is known that the shortest distance between the lines,  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ , is given by,

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|} \quad \dots(1)$$

Comparing the given equations, we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + \hat{k}$$

$$\vec{b}_1 = \hat{i} - \hat{j} + \hat{k}$$

$$\vec{a}_2 = 2\hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_2 = 2\hat{i} + \hat{j} + 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (2\hat{i} - \hat{j} - \hat{k}) - (\hat{i} + 2\hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 2\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 1 \\ 2 & 1 & 2 \end{vmatrix}$$

$$\vec{b}_1 \times \vec{b}_2 = (-2-1)\hat{i} - (2-2)\hat{j} + (1+2)\hat{k} = -3\hat{i} + 3\hat{k}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(-3)^2 + (3)^2} = \sqrt{9+9} = \sqrt{18} = 3\sqrt{2}$$

Substituting all the values in equation (1), we obtain



$$d = \left| \frac{(-3\hat{i} + 3\hat{k}) \cdot (\hat{i} - 3\hat{j} - 2\hat{k})}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-3 \cdot 1 + 3(-2)}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \left| \frac{-9}{3\sqrt{2}} \right|$$

$$\Rightarrow d = \frac{3}{\sqrt{2}} = \frac{3 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} = \frac{3\sqrt{2}}{2}$$

Therefore, the shortest distance between the two lines is  $\frac{3\sqrt{2}}{2}$  units.

**Question 15:**

Find the shortest distance between the lines  $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$  and  $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

Answer

The given lines are  $\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$  and  $\frac{x-3}{1} = \frac{y-5}{-2} = \frac{z-7}{1}$

It is known that the shortest distance between the two lines,

$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$  and  $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ , is given by,

$$d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2}} \quad \dots(1)$$

Comparing the given equations, we obtain

$$x_1 = -1, y_1 = -1, z_1 = -1$$

$$a_1 = 7, b_1 = -6, c_1 = 1$$

$$x_2 = 3, y_2 = 5, z_2 = 7$$

$$a_2 = 1, b_2 = -2, c_2 = 1$$

$$\begin{aligned} \text{Then, } \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} &= \begin{vmatrix} 4 & 6 & 8 \\ 7 & -6 & 1 \\ 1 & -2 & 1 \end{vmatrix} \\ &= 4(-6+2) - 6(7-1) + 8(-14+6) \\ &= -16 - 36 - 64 \\ &= -116 \end{aligned}$$

$$\begin{aligned} \Rightarrow \sqrt{(b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2 + (a_1b_2 - a_2b_1)^2} &= \sqrt{(-6+2)^2 + (1+7)^2 + (-14+6)^2} \\ &= \sqrt{16+36+64} \\ &= \sqrt{116} \\ &= 2\sqrt{29} \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \frac{-116}{2\sqrt{29}} = \frac{-58}{\sqrt{29}} = \frac{-2 \times 29}{\sqrt{29}} = -2\sqrt{29}$$

Since distance is always non-negative, the distance between the given lines is

$$2\sqrt{29} \text{ units.}$$

### Question 16:

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} - 3\hat{j} + 2\hat{k})$$

$$\text{and } \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

Answer

$$\text{The given lines are } \vec{r} = \hat{i} + 2\hat{j} + 3\hat{k} + \lambda(\hat{i} - 3\hat{j} + 2\hat{k}) \text{ and } \vec{r} = 4\hat{i} + 5\hat{j} + 6\hat{k} + \mu(2\hat{i} + 3\hat{j} + \hat{k})$$

It is known that the shortest distance between the lines,  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ , is given by,

$$d = \frac{\left| (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) \right|}{\left| \vec{b}_1 \times \vec{b}_2 \right|} \quad \dots(1)$$

Comparing the given equations with  $\vec{r} = \vec{a}_1 + \lambda \vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu \vec{b}_2$ , we obtain

$$\vec{a}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = \hat{i} - 3\hat{j} + 2\hat{k}$$

$$\vec{a}_2 = 4\hat{i} + 5\hat{j} + 6\hat{k}$$

$$\vec{b}_2 = 2\hat{i} + 3\hat{j} + \hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (4\hat{i} + 5\hat{j} + 6\hat{k}) - (\hat{i} + 2\hat{j} + 3\hat{k}) = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -3 & 2 \\ 2 & 3 & 1 \end{vmatrix} = (-3-6)\hat{i} - (1-4)\hat{j} + (3+6)\hat{k} = -9\hat{i} + 3\hat{j} + 9\hat{k}$$

$$\Rightarrow \left| \vec{b}_1 \times \vec{b}_2 \right| = \sqrt{(-9)^2 + (3)^2 + (9)^2} = \sqrt{81+9+81} = \sqrt{171} = 3\sqrt{19}$$

$$\begin{aligned} (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) &= (-9\hat{i} + 3\hat{j} + 9\hat{k}) \cdot (3\hat{i} + 3\hat{j} + 3\hat{k}) \\ &= -9 \times 3 + 3 \times 3 + 9 \times 3 \\ &= 9 \end{aligned}$$

Substituting all the values in equation (1), we obtain

$$d = \frac{\left| 9 \right|}{\left| 3\sqrt{19} \right|} = \frac{3}{\sqrt{19}}$$

Therefore, the shortest distance between the two given lines is  $\frac{3}{\sqrt{19}}$  units.

### Question 17:

Find the shortest distance between the lines whose vector equations are

$$\vec{r} = (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \text{ and}$$

$$\vec{r} = (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k}$$

Answer

The given lines are

$$\begin{aligned}\vec{r} &= (1-t)\hat{i} + (t-2)\hat{j} + (3-2t)\hat{k} \\ \Rightarrow \vec{r} &= (\hat{i} - 2\hat{j} + 3\hat{k}) + t(-\hat{i} + \hat{j} - 2\hat{k}) \quad \dots(1)\end{aligned}$$

$$\begin{aligned}\vec{r} &= (s+1)\hat{i} + (2s-1)\hat{j} - (2s+1)\hat{k} \\ \Rightarrow \vec{r} &= (\hat{i} - \hat{j} + \hat{k}) + s(\hat{i} + 2\hat{j} - 2\hat{k}) \quad \dots(2)\end{aligned}$$

It is known that the shortest distance between the lines,  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ , is given by,

$$d = \frac{|(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)|}{|\vec{b}_1 \times \vec{b}_2|} \quad \dots(3)$$

For the given equations,

$$\vec{a}_1 = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\vec{b}_1 = -\hat{i} + \hat{j} - 2\hat{k}$$

$$\vec{a}_2 = \hat{i} - \hat{j} - \hat{k}$$

$$\vec{b}_2 = \hat{i} + 2\hat{j} - 2\hat{k}$$

$$\vec{a}_2 - \vec{a}_1 = (\hat{i} - \hat{j} - \hat{k}) - (\hat{i} - 2\hat{j} + 3\hat{k}) = \hat{j} - 4\hat{k}$$

$$\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 1 & -2 \\ 1 & 2 & -2 \end{vmatrix} = (-2+4)\hat{i} - (2+2)\hat{j} + (-2-1)\hat{k} = 2\hat{i} - 4\hat{j} - 3\hat{k}$$

$$\Rightarrow |\vec{b}_1 \times \vec{b}_2| = \sqrt{(2)^2 + (-4)^2 + (-3)^2} = \sqrt{4+16+9} = \sqrt{29}$$

$$\therefore (\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (2\hat{i} - 4\hat{j} - 3\hat{k}) \cdot (\hat{j} - 4\hat{k}) = -4 + 12 = 8$$

Substituting all the values in equation (3), we obtain

$$d = \frac{|8|}{\sqrt{29}} = \frac{8}{\sqrt{29}}$$

Therefore, the shortest distance between the lines is  $\frac{8}{\sqrt{29}}$  units.

## Exercise 11.3

**Question 1:**

In each of the following cases, determine the direction cosines of the normal to the plane and the distance from the origin.

(a)  $z = 2$  (b)  $x + y + z = 1$

(c)  $2x + 3y - z = 5$  (d)  $5y + 8 = 0$

Answer

**(a)** The equation of the plane is  $z = 2$  or  $0x + 0y + z = 2 \dots (1)$

The direction ratios of normal are 0, 0, and 1.

$$\therefore \sqrt{0^2 + 0^2 + 1^2} = 1$$

Dividing both sides of equation (1) by 1, we obtain

$$0.x + 0.y + 1.z = 2$$

This is of the form  $lx + my + nz = d$ , where  $l, m, n$  are the direction cosines of normal to the plane and  $d$  is the distance of the perpendicular drawn from the origin.

Therefore, the direction cosines are 0, 0, and 1 and the distance of the plane from the origin is 2 units.

**(b)**  $x + y + z = 1 \dots (1)$

The direction ratios of normal are 1, 1, and 1.

$$\therefore \sqrt{(1)^2 + (1)^2 + (1)^2} = \sqrt{3}$$

Dividing both sides of equation (1) by  $\sqrt{3}$ , we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}} \dots (2)$$

This equation is of the form  $lx + my + nz = d$ , where  $l, m, n$  are the direction cosines of normal to the plane and  $d$  is the distance of normal from the origin.

Therefore, the direction cosines of the normal are  $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ , and  $\frac{1}{\sqrt{3}}$  and the distance of normal from the origin is  $\frac{1}{\sqrt{3}}$  units.

**(c)**  $2x + 3y - z = 5 \dots (1)$

The direction ratios of normal are 2, 3, and  $-1$ .

$$\therefore \sqrt{(2)^2 + (3)^2 + (-1)^2} = \sqrt{14}$$

Dividing both sides of equation (1) by  $\sqrt{14}$ , we obtain

$$\frac{2}{\sqrt{14}}x + \frac{3}{\sqrt{14}}y - \frac{1}{\sqrt{14}}z = \frac{5}{\sqrt{14}}$$

This equation is of the form  $lx + my + nz = d$ , where  $l, m, n$  are the direction cosines of normal to the plane and  $d$  is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are  $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$ , and  $\frac{-1}{\sqrt{14}}$  and

the distance of normal from the origin is  $\frac{5}{\sqrt{14}}$  units.

**(d)**  $5y + 8 = 0$

$$\Rightarrow 0x - 5y + 0z = 8 \dots (1)$$

The direction ratios of normal are 0,  $-5$ , and 0.

$$\therefore \sqrt{0 + (-5)^2 + 0} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y = \frac{8}{5}$$

This equation is of the form  $lx + my + nz = d$ , where  $l, m, n$  are the direction cosines of normal to the plane and  $d$  is the distance of normal from the origin.

Therefore, the direction cosines of the normal to the plane are  $0, -1,$  and  $0$  and the

distance of normal from the origin is  $\frac{8}{5}$  units.

### Question 2:

Find the vector equation of a plane which is at a distance of 7 units from the origin and normal to the vector  $3\hat{i} + 5\hat{j} - 6\hat{k}$ .

Answer

The normal vector is,  $\vec{n} = 3\hat{i} + 5\hat{j} - 6\hat{k}$

$$\therefore \hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{(3)^2 + (5)^2 + (6)^2}} = \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}}$$

It is known that the equation of the plane with position vector  $\vec{r}$  is given by,  $\vec{r} \cdot \hat{n} = d$

$$\Rightarrow \vec{r} \cdot \left( \frac{3\hat{i} + 5\hat{j} - 6\hat{k}}{\sqrt{70}} \right) = 7$$

This is the vector equation of the required plane.

### Question 3:

Find the Cartesian equation of the following planes:

(a)  $\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2$       (b)  $\vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1$

(c)  $\vec{r} \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] = 15$

Answer

**(a)** It is given that equation of the plane is

$$\vec{r} \cdot (\hat{i} + \hat{j} - \hat{k}) = 2 \quad \dots(1)$$

For any arbitrary point P ( $x, y, z$ ) on the plane, position vector  $\vec{r}$  is given by,

$$\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$$

Substituting the value of  $\vec{r}$  in equation (1), we obtain

$$\begin{aligned} (x\hat{i} + y\hat{j} - z\hat{k}) \cdot (\hat{i} + \hat{j} - \hat{k}) &= 2 \\ \Rightarrow x + y - z &= 2 \end{aligned}$$

This is the Cartesian equation of the plane.

$$\text{(b)} \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) = 1 \quad \dots(1)$$

For any arbitrary point P (x, y, z) on the plane, position vector  $\vec{r}$  is given by,

$$\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$$

Substituting the value of  $\vec{r}$  in equation (1), we obtain

$$\begin{aligned} (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (2\hat{i} + 3\hat{j} - 4\hat{k}) &= 1 \\ \Rightarrow 2x + 3y - 4z &= 1 \end{aligned}$$

This is the Cartesian equation of the plane.

$$\text{(c)} \quad \vec{r} \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] = 15 \quad \dots(1)$$

For any arbitrary point P (x, y, z) on the plane, position vector  $\vec{r}$  is given by,

$$\vec{r} = x\hat{i} + y\hat{j} - z\hat{k}$$

Substituting the value of  $\vec{r}$  in equation (1), we obtain

$$\begin{aligned} (x\hat{i} + y\hat{j} - z\hat{k}) \cdot [(s-2t)\hat{i} + (3-t)\hat{j} + (2s+t)\hat{k}] &= 15 \\ \Rightarrow (s-2t)x + (3-t)y + (2s+t)z &= 15 \end{aligned}$$

This is the Cartesian equation of the given plane.

#### Question 4:

In the following cases, find the coordinates of the foot of the perpendicular drawn from the origin.

$$\text{(a)} \quad 2x + 3y + 4z - 12 = 0 \quad \text{(b)} \quad 3y + 4z - 6 = 0$$

$$\text{(c)} \quad x + y + z = 1 \quad \text{(d)} \quad 5y + 8 = 0$$

Answer

**(a)** Let the coordinates of the foot of perpendicular P from the origin to the plane be  $(x_1, y_1, z_1)$ .



$$2x + 3y + 4z - 12 = 0$$

$$\Rightarrow 2x + 3y + 4z = 12 \dots (1)$$

The direction ratios of normal are 2, 3, and 4.

$$\therefore \sqrt{(2)^2 + (3)^2 + (4)^2} = \sqrt{29}$$

Dividing both sides of equation (1) by  $\sqrt{29}$ , we obtain

$$\frac{2}{\sqrt{29}}x + \frac{3}{\sqrt{29}}y + \frac{4}{\sqrt{29}}z = \frac{12}{\sqrt{29}}$$

This equation is of the form  $lx + my + nz = d$ , where  $l, m, n$  are the direction cosines of normal to the plane and  $d$  is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by  $(ld, md, nd)$ .

Therefore, the coordinates of the foot of the perpendicular are

$$\left( \frac{2}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}}, \frac{3}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}}, \frac{4}{\sqrt{29}} \cdot \frac{12}{\sqrt{29}} \right) \text{ i.e., } \left( \frac{24}{29}, \frac{36}{29}, \frac{48}{29} \right).$$

**(b)** Let the coordinates of the foot of perpendicular P from the origin to the plane be  $(x_1, y_1, z_1)$ .

$$3y + 4z - 6 = 0$$

$$\Rightarrow 0x + 3y + 4z = 6 \dots (1)$$

The direction ratios of the normal are 0, 3, and 4.

$$\therefore \sqrt{0+3^2+4^2} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$0x + \frac{3}{5}y + \frac{4}{5}z = \frac{6}{5}$$

This equation is of the form  $lx + my + nz = d$ , where  $l, m, n$  are the direction cosines of normal to the plane and  $d$  is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by  $(ld, md, nd)$ .

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, \frac{3}{5} \cdot \frac{6}{5}, \frac{4}{5} \cdot \frac{6}{5}\right) \text{ i.e., } \left(0, \frac{18}{25}, \frac{24}{25}\right).$$

**(c)** Let the coordinates of the foot of perpendicular P from the origin to the plane be  $(x_1, y_1, z_1)$ .

$$x + y + z = 1 \dots (1)$$

The direction ratios of the normal are 1, 1, and 1.

$$\therefore \sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$$

Dividing both sides of equation (1) by  $\sqrt{3}$ , we obtain

$$\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z = \frac{1}{\sqrt{3}}$$

This equation is of the form  $lx + my + nz = d$ , where  $l, m, n$  are the direction cosines of normal to the plane and  $d$  is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by  $(ld, md, nd)$ .

Therefore, the coordinates of the foot of the perpendicular are

$$\left(\frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \cdot \frac{1}{\sqrt{3}}\right) \text{ i.e., } \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right).$$

**(d)** Let the coordinates of the foot of perpendicular P from the origin to the plane be  $(x_1, y_1, z_1)$ .

$$5y + 8 = 0$$

$$\Rightarrow 0x - 5y + 0z = 8 \dots (1)$$

The direction ratios of the normal are 0, -5, and 0.

$$\therefore \sqrt{0+(-5)^2+0} = 5$$

Dividing both sides of equation (1) by 5, we obtain

$$-y = \frac{8}{5}$$

This equation is of the form  $lx + my + nz = d$ , where  $l, m, n$  are the direction cosines of normal to the plane and  $d$  is the distance of normal from the origin.

The coordinates of the foot of the perpendicular are given by  $(ld, md, nd)$ .

Therefore, the coordinates of the foot of the perpendicular are

$$\left(0, -1\left(\frac{8}{5}\right), 0\right) \text{ i.e., } \left(0, -\frac{8}{5}, 0\right).$$

#### Question 5:

Find the vector and Cartesian equation of the planes

(a) that passes through the point  $(1, 0, -2)$  and the normal to the plane is  $\hat{i} + \hat{j} - \hat{k}$ .

(b) that passes through the point  $(1, 4, 6)$  and the normal vector to the plane is

$$\hat{i} - 2\hat{j} + \hat{k}.$$

Answer

**(a)** The position vector of point  $(1, 0, -2)$  is  $\vec{a} = \hat{i} - 2\hat{k}$

The normal vector  $\vec{N}$  perpendicular to the plane is  $\vec{N} = \hat{i} + \hat{j} - \hat{k}$

The vector equation of the plane is given by,  $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$$\Rightarrow [\vec{r} - (\hat{i} - 2\hat{k})] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \quad \dots(1)$$

$\vec{r}$  is the position vector of any point P  $(x, y, z)$  in the plane.

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Therefore, equation (1) becomes

$$\begin{aligned} & \left[ (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} - 2\hat{k}) \right] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \\ \Rightarrow & \left[ (x-1)\hat{i} + y\hat{j} + (z+2)\hat{k} \right] \cdot (\hat{i} + \hat{j} - \hat{k}) = 0 \\ \Rightarrow & (x-1) + y - (z+2) = 0 \\ \Rightarrow & x + y - z - 3 = 0 \\ \Rightarrow & x + y - z = 3 \end{aligned}$$

This is the Cartesian equation of the required plane.

**(b)** The position vector of the point (1, 4, 6) is  $\vec{a} = \hat{i} + 4\hat{j} + 6\hat{k}$

The normal vector  $\vec{N}$  perpendicular to the plane is  $\vec{N} = \hat{i} - 2\hat{j} + \hat{k}$

The vector equation of the plane is given by,  $(\vec{r} - \vec{a}) \cdot \vec{N} = 0$

$$\Rightarrow \left[ \vec{r} - (\hat{i} + 4\hat{j} + 6\hat{k}) \right] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \quad \dots(1)$$

$\vec{r}$  is the position vector of any point P (x, y, z) in the plane.

$$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Therefore, equation (1) becomes

$$\begin{aligned} & \left[ (x\hat{i} + y\hat{j} + z\hat{k}) - (\hat{i} + 4\hat{j} + 6\hat{k}) \right] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \\ \Rightarrow & \left[ (x-1)\hat{i} + (y-4)\hat{j} + (z-6)\hat{k} \right] \cdot (\hat{i} - 2\hat{j} + \hat{k}) = 0 \\ \Rightarrow & (x-1) - 2(y-4) + (z-6) = 0 \\ \Rightarrow & x - 2y + z + 1 = 0 \end{aligned}$$

This is the Cartesian equation of the required plane.

### Question 6:

Find the equations of the planes that passes through three points.

(a) (1, 1, -1), (6, 4, -5), (-4, -2, 3)

(b) (1, 1, 0), (1, 2, 1), (-2, 2, -1)

Answer

**(a)** The given points are A (1, 1, -1), B (6, 4, -5), and C (-4, -2, 3).

$$\begin{vmatrix} 1 & 1 & -1 \\ 6 & 4 & -5 \\ -4 & -2 & 3 \end{vmatrix} = (12 - 10) - (18 - 20) - (-12 + 16) \\ = 2 + 2 - 4 \\ = 0$$

Since A, B, C are collinear points, there will be infinite number of planes passing through the given points.

**(b)** The given points are A (1, 1, 0), B (1, 2, 1), and C (-2, 2, -1).

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ -2 & 2 & -1 \end{vmatrix} = (-2 - 2) - (2 + 2) = -8 \neq 0$$

Therefore, a plane will pass through the points A, B, and C.

It is known that the equation of the plane through the points,  $(x_1, y_1, z_1)$ ,  $(x_2, y_2, z_2)$ , and  $(x_3, y_3, z_3)$ , is

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0 \\ \Rightarrow \begin{vmatrix} x - 1 & y - 1 & z \\ 0 & 1 & 1 \\ -3 & 1 & -1 \end{vmatrix} = 0 \\ \Rightarrow (-2)(x - 1) - 3(y - 1) + 3z = 0 \\ \Rightarrow -2x - 3y + 3z + 2 + 3 = 0 \\ \Rightarrow -2x - 3y + 3z = -5 \\ \Rightarrow 2x + 3y - 3z = 5$$

This is the Cartesian equation of the required plane.

### Question 7:

Find the intercepts cut off by the plane  $2x + y - z = 5$

Answer

$$2x + y - z = 5 \quad \dots(1)$$

Dividing both sides of equation (1) by 5, we obtain

$$\frac{2}{5}x + \frac{y}{5} - \frac{z}{5} = 1$$

$$\Rightarrow \frac{x}{\frac{5}{2}} + \frac{y}{5} + \frac{z}{-5} = 1 \quad \dots(2)$$

It is known that the equation of a plane in intercept form is  $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ , where  $a, b, c$  are the intercepts cut off by the plane at  $x, y,$  and  $z$  axes respectively.

Therefore, for the given equation,

$$a = \frac{5}{2}, b = 5, \text{ and } c = -5$$

Thus, the intercepts cut off by the plane are  $\frac{5}{2}, 5,$  and  $-5$ .

**Question 8:**

Find the equation of the plane with intercept 3 on the  $y$ -axis and parallel to  $ZOX$  plane.

Answer

The equation of the plane  $ZOX$  is

$$y = 0$$

Any plane parallel to it is of the form,  $y = a$

Since the  $y$ -intercept of the plane is 3,

$$\therefore a = 3$$

Thus, the equation of the required plane is  $y = 3$

**Question 9:**

Find the equation of the plane through the intersection of the planes

$$3x - y + 2z - 4 = 0 \text{ and } x + y + z - 2 = 0 \text{ and the point } (2, 2, 1)$$

Answer

The equation of any plane through the intersection of the planes,

$3x - y + 2z - 4 = 0$  and  $x + y + z - 2 = 0$ , is

$$(3x - y + 2z - 4) + \alpha(x + y + z - 2) = 0, \text{ where } \alpha \in \mathbb{R} \quad \dots(1)$$

The plane passes through the point  $(2, 2, 1)$ . Therefore, this point will satisfy equation (1).

$$\therefore (3 \times 2 - 2 + 2 \times 1 - 4) + \alpha(2 + 2 + 1 - 2) = 0$$

$$\Rightarrow 2 + 3\alpha = 0$$

$$\Rightarrow \alpha = -\frac{2}{3}$$

Substituting  $\alpha = -\frac{2}{3}$  in equation (1), we obtain

$$(3x - y + 2z - 4) - \frac{2}{3}(x + y + z - 2) = 0$$

$$\Rightarrow 3(3x - y + 2z - 4) - 2(x + y + z - 2) = 0$$

$$\Rightarrow (9x - 3y + 6z - 12) - 2(x + y + z - 2) = 0$$

$$\Rightarrow 7x - 5y + 4z - 8 = 0$$

This is the required equation of the plane.

### Question 10:

Find the vector equation of the plane passing through the intersection of the planes

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7, \quad \vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$$

and through the point  $(2, 1, 3)$

Answer

The equations of the planes are  $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 7$  and  $\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) = 9$

$$\Rightarrow \vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) - 7 = 0 \quad \dots(1)$$

$$\vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) - 9 = 0 \quad \dots(2)$$

The equation of any plane through the intersection of the planes given in equations (1) and (2) is given by,

$$\left[ \vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) - 7 \right] + \lambda \left[ \vec{r} \cdot (2\hat{i} + 5\hat{j} + 3\hat{k}) - 9 \right] = 0, \text{ where } \lambda \in \mathbb{R}$$

$$\vec{r} \cdot [(2\hat{i} + 2\hat{j} - 3\hat{k}) + \lambda(2\hat{i} + 5\hat{j} + 3\hat{k})] = 9\lambda + 7$$

$$\vec{r} \cdot [(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}] = 9\lambda + 7 \quad \dots(3)$$

The plane passes through the point (2, 1, 3). Therefore, its position vector is given by,

$$\vec{r} = 2\hat{i} + \hat{j} + 3\hat{k}$$

Substituting in equation (3), we obtain

$$(2\hat{i} + \hat{j} - 3\hat{k}) \cdot [(2 + 2\lambda)\hat{i} + (2 + 5\lambda)\hat{j} + (3\lambda - 3)\hat{k}] = 9\lambda + 7$$

$$\Rightarrow (2 + 2\lambda) + (2 + 5\lambda) + (3\lambda - 3) = 9\lambda + 7$$

$$\Rightarrow 18\lambda - 3 = 9\lambda + 7$$

$$\Rightarrow 9\lambda = 10$$

$$\Rightarrow \lambda = \frac{10}{9}$$

Substituting  $\lambda = \frac{10}{9}$  in equation (3), we obtain

$$\vec{r} \cdot \left( \frac{38}{9}\hat{i} + \frac{68}{9}\hat{j} + \frac{3}{9}\hat{k} \right) = 17$$

$$\Rightarrow \vec{r} \cdot (38\hat{i} + 68\hat{j} + 3\hat{k}) = 153$$

This is the vector equation of the required plane.

### Question 11:

Find the equation of the plane through the line of intersection of the planes

$$x + y + z = 1 \text{ and } 2x + 3y + 4z = 5 \text{ which is perpendicular to the plane } x - y + z = 0$$

Answer

The equation of the plane through the intersection of the planes,  $x + y + z = 1$  and  $2x + 3y + 4z = 5$ , is

$$(x + y + z - 1) + \lambda(2x + 3y + 4z - 5) = 0$$

$$\Rightarrow (2\lambda + 1)x + (3\lambda + 1)y + (4\lambda + 1)z - (5\lambda + 1) = 0 \quad \dots(1)$$

The direction ratios,  $a_1, b_1, c_1$ , of this plane are  $(2\lambda + 1)$ ,  $(3\lambda + 1)$ , and  $(4\lambda + 1)$ .

The plane in equation (1) is perpendicular to  $x - y + z = 0$



Its direction ratios,  $a_2, b_2, c_2$ , are 1, -1, and 1.

Since the planes are perpendicular,

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

$$\Rightarrow (2\lambda + 1) - (3\lambda + 1) + (4\lambda + 1) = 0$$

$$\Rightarrow 3\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{3}$$

Substituting  $\lambda = -\frac{1}{3}$  in equation (1), we obtain

$$\frac{1}{3}x - \frac{1}{3}z + \frac{2}{3} = 0$$

$$\Rightarrow x - z + 2 = 0$$

This is the required equation of the plane.

### Question 12:

Find the angle between the planes whose vector equations are

$$\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5 \quad \text{and} \quad \vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$$

Answer

The equations of the given planes are  $\vec{r} \cdot (2\hat{i} + 2\hat{j} - 3\hat{k}) = 5$  and  $\vec{r} \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 3$

It is known that if  $\vec{n}_1$  and  $\vec{n}_2$  are normal to the planes,  $\vec{r} \cdot \vec{n}_1 = d_1$  and  $\vec{r} \cdot \vec{n}_2 = d_2$ , then the angle between them,  $Q$ , is given by,

$$\cos Q = \frac{|\vec{n}_1 \cdot \vec{n}_2|}{|\vec{n}_1| |\vec{n}_2|} \quad \dots(1)$$

Here,  $\vec{n}_1 = 2\hat{i} + 2\hat{j} - 3\hat{k}$  and  $\vec{n}_2 = 3\hat{i} - 3\hat{j} + 5\hat{k}$

$$\therefore \vec{n}_1 \cdot \vec{n}_2 = (2\hat{i} + 2\hat{j} - 3\hat{k}) \cdot (3\hat{i} - 3\hat{j} + 5\hat{k}) = 2 \cdot 3 + 2 \cdot (-3) + (-3) \cdot 5 = -15$$

$$|\vec{n}_1| = \sqrt{(2)^2 + (2)^2 + (-3)^2} = \sqrt{17}$$

$$|\vec{n}_2| = \sqrt{(3)^2 + (-3)^2 + (5)^2} = \sqrt{43}$$

Substituting the value of  $\vec{n} \cdot \vec{n}_2$ ,  $|\vec{n}_1|$  and  $|\vec{n}_2|$  in equation (1), we obtain

$$\cos Q = \left| \frac{-15}{\sqrt{17} \cdot \sqrt{43}} \right|$$

$$\Rightarrow \cos Q = \frac{15}{\sqrt{731}}$$

$$\Rightarrow \cos Q^{-1} = \left( \frac{15}{\sqrt{731}} \right)$$

### Question 13:

In the following cases, determine whether the given planes are parallel or perpendicular, and in case they are neither, find the angles between them.

(a)  $7x + 5y + 6z + 30 = 0$  and  $3x - y - 10z + 4 = 0$

(b)  $2x + y + 3z - 2 = 0$  and  $x - 2y + 5 = 0$

(c)  $2x - 2y + 4z + 5 = 0$  and  $3x - 3y + 6z - 1 = 0$

(d)  $2x - y + 3z - 1 = 0$  and  $2x - y + 3z + 3 = 0$

(e)  $4x + 8y + z - 8 = 0$  and  $y + z - 4 = 0$

Answer

The direction ratios of normal to the plane,  $L_1 : a_1x + b_1y + c_1z = 0$ , are  $a_1, b_1, c_1$  and  $L_2 : a_2x + b_2y + c_2z = 0$  are  $a_2, b_2, c_2$ .

$$L_1 \parallel L_2, \text{ if } \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

$$L_1 \perp L_2, \text{ if } a_1a_2 + b_1b_2 + c_1c_2 = 0$$

The angle between  $L_1$  and  $L_2$  is given by,

$$Q = \cos^{-1} \left| \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \cdot \sqrt{a_2^2 + b_2^2 + c_2^2}} \right|$$

(a) The equations of the planes are  $7x + 5y + 6z + 30 = 0$  and

$$3x - y - 10z + 4 = 0$$

Here,  $a_1 = 7, b_1 = 5, c_1 = 6$

$$a_2 = 3, b_2 = -1, c_2 = -10$$

$$a_1a_2 + b_1b_2 + c_1c_2 = 7 \times 3 + 5 \times (-1) + 6 \times (-10) = -44 \neq 0$$

Therefore, the given planes are not perpendicular.

$$\frac{a_1}{a_2} = \frac{7}{3}, \frac{b_1}{b_2} = \frac{5}{-1} = -5, \frac{c_1}{c_2} = \frac{6}{-10} = \frac{-3}{5}$$

It can be seen that,  $\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$

Therefore, the given planes are not parallel.

The angle between them is given by,

$$\begin{aligned} Q &= \cos^{-1} \left| \frac{7 \times 3 + 5 \times (-1) + 6 \times (-10)}{\sqrt{(7)^2 + (5)^2 + (6)^2} \times \sqrt{(3)^2 + (-1)^2 + (-10)^2}} \right| \\ &= \cos^{-1} \left| \frac{21 - 5 - 60}{\sqrt{110} \times \sqrt{110}} \right| \\ &= \cos^{-1} \frac{44}{110} \\ &= \cos^{-1} \frac{2}{5} \end{aligned}$$

**(b)** The equations of the planes are  $2x + y + 3z - 2 = 0$  and  $x - 2y + 5 = 0$

Here,  $a_1 = 2, b_1 = 1, c_1 = 3$  and  $a_2 = 1, b_2 = -2, c_2 = 0$

$$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 1 \times (-2) + 3 \times 0 = 0$$

Thus, the given planes are perpendicular to each other.

**(c)** The equations of the given planes are  $2x - 2y + 4z + 5 = 0$  and  $3x - 3y + 6z - 1 = 0$

Here,  $a_1 = 2, b_1 = -2, c_1 = 4$  and

$$a_2 = 3, b_2 = -3, c_2 = 6 \quad a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 3 + (-2)(-3) + 4 \times 6 = 6 + 6 + 24 = 36 \neq 0$$

Thus, the given planes are not perpendicular to each other.

$$\frac{a_1}{a_2} = \frac{2}{3}, \frac{b_1}{b_2} = \frac{-2}{-3} = \frac{2}{3} \text{ and } \frac{c_1}{c_2} = \frac{4}{6} = \frac{2}{3}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the given planes are parallel to each other.

**(d)** The equations of the planes are  $2x - y + 3z - 1 = 0$  and  $2x - y + 3z + 3 = 0$

Here,  $a_1 = 2, b_1 = -1, c_1 = 3$  and  $a_2 = 2, b_2 = -1, c_2 = 3$

$$\frac{a_1}{a_2} = \frac{2}{2} = 1, \frac{b_1}{b_2} = \frac{-1}{-1} = 1 \text{ and } \frac{c_1}{c_2} = \frac{3}{3} = 1$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Thus, the given lines are parallel to each other.

**(e)** The equations of the given planes are  $4x + 8y + z - 8 = 0$  and  $y + z - 4 = 0$

Here,  $a_1 = 4, b_1 = 8, c_1 = 1$  and  $a_2 = 0, b_2 = 1, c_2 = 1$

$$a_1 a_2 + b_1 b_2 + c_1 c_2 = 4 \times 0 + 8 \times 1 + 1 = 9 \neq 0$$

Therefore, the given lines are not perpendicular to each other.

$$\frac{a_1}{a_2} = \frac{4}{0}, \frac{b_1}{b_2} = \frac{8}{1} = 8, \frac{c_1}{c_2} = \frac{1}{1} = 1$$

$$\therefore \frac{a_1}{a_2} \neq \frac{b_1}{b_2} \neq \frac{c_1}{c_2}$$

Therefore, the given lines are not parallel to each other.

The angle between the planes is given by,

$$Q = \cos^{-1} \left| \frac{4 \times 0 + 8 \times 1 + 1 \times 1}{\sqrt{4^2 + 8^2 + 1^2} \times \sqrt{0^2 + 1^2 + 1^2}} \right| = \cos^{-1} \left| \frac{9}{9 \times \sqrt{2}} \right| = \cos^{-1} \left( \frac{1}{\sqrt{2}} \right) = 45^\circ$$

**Question 14:**

In the following cases, find the distance of each of the given points from the corresponding given plane.

**Point Plane**

(a)  $(0, 0, 0)$   $3x - 4y + 12z = 3$

(b)  $(3, -2, 1)$   $2x - y + 2z + 3 = 0$

(c)  $(2, 3, -5)$   $x + 2y - 2z = 9$

(d)  $(-6, 0, 0)$   $2x - 3y + 6z - 2 = 0$

Answer

It is known that the distance between a point,  $p(x_1, y_1, z_1)$ , and a plane,  $Ax + By + Cz = D$ , is given by,

$$d = \left| \frac{Ax_1 + By_1 + Cz_1 - D}{\sqrt{A^2 + B^2 + C^2}} \right| \quad \dots(1)$$

**(a)** The given point is  $(0, 0, 0)$  and the plane is  $3x - 4y + 12z = 3$

$$\therefore d = \left| \frac{3 \times 0 - 4 \times 0 + 12 \times 0 - 3}{\sqrt{(3)^2 + (-4)^2 + (12)^2}} \right| = \frac{3}{\sqrt{169}} = \frac{3}{13}$$

**(b)** The given point is  $(3, -2, 1)$  and the plane is  $2x - y + 2z + 3 = 0$

$$\therefore d = \left| \frac{2 \times 3 - (-2) + 2 \times 1 + 3}{\sqrt{(2)^2 + (-1)^2 + (2)^2}} \right| = \left| \frac{13}{3} \right| = \frac{13}{3}$$

**(c)** The given point is  $(2, 3, -5)$  and the plane is  $x + 2y - 2z = 9$

$$\therefore d = \left| \frac{2 + 2 \times 3 - 2(-5) - 9}{\sqrt{(1)^2 + (2)^2 + (-2)^2}} \right| = \frac{9}{3} = 3$$

**(d)** The given point is  $(-6, 0, 0)$  and the plane is  $2x - 3y + 6z - 2 = 0$

$$d = \left| \frac{2(-6) - 3 \times 0 + 6 \times 0 - 2}{\sqrt{(2)^2 + (-3)^2 + (6)^2}} \right| = \left| \frac{-14}{\sqrt{49}} \right| = \frac{14}{7} = 2$$

## Miscellaneous Solutions

**Question 1:**

Show that the line joining the origin to the point  $(2, 1, 1)$  is perpendicular to the line determined by the points  $(3, 5, -1), (4, 3, -1)$ .

Answer

Let OA be the line joining the origin, O  $(0, 0, 0)$ , and the point, A  $(2, 1, 1)$ .

Also, let BC be the line joining the points, B  $(3, 5, -1)$  and C  $(4, 3, -1)$ .

The direction ratios of OA are 2, 1, and 1 and of BC are  $(4 - 3) = 1, (3 - 5) = -2$ , and  $(-1 + 1) = 0$

OA is perpendicular to BC, if  $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\therefore a_1a_2 + b_1b_2 + c_1c_2 = 2 \times 1 + 1(-2) + 1 \times 0 = 2 - 2 = 0$$

Thus, OA is perpendicular to BC.

**Question 2:**

If  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$  are the direction cosines of two mutually perpendicular lines, show that the direction cosines of the line perpendicular to both of these are  $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1$ .

Answer

It is given that  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$  are the direction cosines of two mutually perpendicular lines. Therefore,

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \quad \dots(1)$$

$$l_1^2 + m_1^2 + n_1^2 = 1 \quad \dots(2)$$

$$l_2^2 + m_2^2 + n_2^2 = 1 \quad \dots(3)$$

Let  $l, m, n$  be the direction cosines of the line which is perpendicular to the line with direction cosines  $l_1, m_1, n_1$  and  $l_2, m_2, n_2$ .

$$\begin{aligned}
 \therefore ll_1 + mm_1 + nn_1 &= 0 \\
 ll_2 + mm_2 + nn_2 &= 0 \\
 \therefore \frac{l}{m_1n_2 - m_2n_1} &= \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} \\
 \Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} &= \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2} \\
 \Rightarrow \frac{l^2}{(m_1n_2 - m_2n_1)^2} &= \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_2)^2} \\
 &= \frac{l^2 + m^2 + n^2}{(m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2} \quad \dots(4)
 \end{aligned}$$

$l, m, n$  are the direction cosines of the line.

$$\therefore l^2 + m^2 + n^2 = 1 \quad \dots (5)$$

It is known that,

$$\begin{aligned}
 (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (ll_2 + m_1m_2 + n_1n_2)^2 \\
 = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2
 \end{aligned}$$

From (1), (2), and (3), we obtain

$$\Rightarrow 1 \cdot 1 - 0 = (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2$$

$$\therefore (m_1n_2 - m_2n_1)^2 + (n_1l_2 - n_2l_1)^2 + (l_1m_2 - l_2m_1)^2 = 1 \quad \dots(6)$$

Substituting the values from equations (5) and (6) in equation (4), we obtain

$$\frac{l^2}{(m_1n_2 - m_2n_1)^2} = \frac{m^2}{(n_1l_2 - n_2l_1)^2} = \frac{n^2}{(l_1m_2 - l_2m_1)^2} = 1$$

$$\Rightarrow l = m_1n_2 - m_2n_1, m = n_1l_2 - n_2l_1, n = l_1m_2 - l_2m_1$$

Thus, the direction cosines of the required line are  $m_1n_2 - m_2n_1, n_1l_2 - n_2l_1,$  and  $l_1m_2 - l_2m_1.$



**Question 3:**

Find the angle between the lines whose direction ratios are  $a, b, c$  and  $b - c, c - a, a - b$ .

Answer

The angle  $Q$  between the lines with direction cosines,  $a, b, c$  and  $b - c, c - a, a - b$ , is given by,

$$\cos Q = \left| \frac{a(b-c) + b(c-a) + c(a-b)}{\sqrt{a^2 + b^2 + c^2} + \sqrt{(b-c)^2 + (c-a)^2 + (a-b)^2}} \right|$$

$$\Rightarrow \cos Q = 0$$

$$\Rightarrow Q = \cos^{-1} 0$$

$$\Rightarrow Q = 90^\circ$$

Thus, the angle between the lines is  $90^\circ$ .

**Question 4:**

Find the equation of a line parallel to x-axis and passing through the origin.

Answer

The line parallel to x-axis and passing through the origin is x-axis itself.

Let A be a point on x-axis. Therefore, the coordinates of A are given by  $(a, 0, 0)$ , where

$$a \in \mathbb{R}.$$

Direction ratios of OA are  $(a - 0) = a, 0, 0$

The equation of OA is given by,

$$\frac{x-0}{a} = \frac{y-0}{0} = \frac{z-0}{0}$$

$$\Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z}{0} = a$$

Thus, the equation of line parallel to x-axis and passing through origin is

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$$

**Question 5:**

If the coordinates of the points A, B, C, D be (1, 2, 3), (4, 5, 7), (-4, 3, -6) and (2, 9, 2) respectively, then find the angle between the lines AB and CD.

Answer

The coordinates of A, B, C, and D are (1, 2, 3), (4, 5, 7), (-4, 3, -6), and (2, 9, 2) respectively.

The direction ratios of AB are  $(4 - 1) = 3$ ,  $(5 - 2) = 3$ , and  $(7 - 3) = 4$

The direction ratios of CD are  $(2 - (-4)) = 6$ ,  $(9 - 3) = 6$ , and  $(2 - (-6)) = 8$

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{1}{2}$$

It can be seen that,

Therefore, AB is parallel to CD.

Thus, the angle between AB and CD is either  $0^\circ$  or  $180^\circ$ .

**sQuestion 6:**

If the lines  $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$  and  $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$  are perpendicular, find the value of  $k$ .

Answer

The direction of ratios of the lines,  $\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$  and  $\frac{x-1}{3k} = \frac{y-1}{1} = \frac{z-6}{-5}$ , are  $-3$ ,  $2k$ ,  $2$  and  $3k$ ,  $1$ ,  $-5$  respectively.

It is known that two lines with direction ratios,  $a_1, b_1, c_1$  and  $a_2, b_2, c_2$ , are perpendicular, if  $a_1a_2 + b_1b_2 + c_1c_2 = 0$

$$\therefore -3(3k) + 2k \times 1 + 2(-5) = 0$$

$$\Rightarrow -9k + 2k - 10 = 0$$

$$\Rightarrow 7k = -10$$

$$\Rightarrow k = \frac{-10}{7}$$

Therefore, for  $k = -\frac{10}{7}$ , the given lines are perpendicular to each other.

**Question 7:**

Find the vector equation of the plane passing through (1, 2, 3) and perpendicular to the

$$\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$$

plane

Answer

The position vector of the point (1, 2, 3) is  $\vec{r}_1 = \hat{i} + 2\hat{j} + 3\hat{k}$

The direction ratios of the normal to the plane,  $\vec{r} \cdot (\hat{i} + 2\hat{j} - 5\hat{k}) + 9 = 0$ , are 1, 2, and -5

and the normal vector is  $\vec{N} = \hat{i} + 2\hat{j} - 5\hat{k}$

The equation of a line passing through a point and perpendicular to the given plane is

given by,  $\vec{l} = \vec{r} + \lambda \vec{N}$ ,  $\lambda \in R$

$$\Rightarrow \vec{l} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(\hat{i} + 2\hat{j} - 5\hat{k})$$

**Question 8:**

Find the equation of the plane passing through (a, b, c) and parallel to the plane

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$$

Answer

Any plane parallel to the plane,  $\vec{r}_1 \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$ , is of the form

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = \lambda \quad \dots(1)$$

The plane passes through the point (a, b, c). Therefore, the position vector  $\vec{r}$  of this

point is  $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$

Therefore, equation (1) becomes

$$(a\hat{i} + b\hat{j} + c\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) = \lambda$$

$$\Rightarrow a + b + c = \lambda$$

Substituting  $\lambda = a + b + c$  in equation (1), we obtain

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c \quad \dots(2)$$

This is the vector equation of the required plane.

Substituting  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  in equation (2), we obtain

$$\begin{aligned} (x\hat{i} + y\hat{j} + z\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k}) &= a + b + c \\ \Rightarrow x + y + z &= a + b + c \end{aligned}$$

### Question 9:

Find the shortest distance between lines  $\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k})$

and  $\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k})$ .

Answer

The given lines are

$$\vec{r} = 6\hat{i} + 2\hat{j} + 2\hat{k} + \lambda(\hat{i} - 2\hat{j} + 2\hat{k}) \quad \dots(1)$$

$$\vec{r} = -4\hat{i} - \hat{k} + \mu(3\hat{i} - 2\hat{j} - 2\hat{k}) \quad \dots(2)$$

It is known that the shortest distance between two lines,  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$ , is given by

$$d = \left| \frac{(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1)}{|\vec{b}_1 \times \vec{b}_2|} \right| \quad \dots(3)$$

Comparing  $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$  and  $\vec{r} = \vec{a}_2 + \lambda\vec{b}_2$  to equations (1) and (2), we obtain

$$\vec{a}_1 = 6\hat{i} + 2\hat{j} + 2\hat{k}$$

$$\vec{b}_1 = \hat{i} - 2\hat{j} + 2\hat{k}$$

$$\vec{a}_2 = -4\hat{i} - \hat{k}$$

$$\vec{b}_2 = 3\hat{i} - 2\hat{j} - 2\hat{k}$$

$$\Rightarrow \vec{a}_2 - \vec{a}_1 = (-4\hat{i} - \hat{k}) - (6\hat{i} + 2\hat{j} + 2\hat{k}) = -10\hat{i} - 2\hat{j} - 3\hat{k}$$

$$\Rightarrow \vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = (4+4)\hat{i} - (-2-6)\hat{j} + (-2+6)\hat{k} = 8\hat{i} + 8\hat{j} + 4\hat{k}$$

$$\therefore |\vec{b}_1 \times \vec{b}_2| = \sqrt{(8)^2 + (8)^2 + (4)^2} = 12$$

$$(\vec{b}_1 \times \vec{b}_2) \cdot (\vec{a}_2 - \vec{a}_1) = (8\hat{i} + 8\hat{j} + 4\hat{k}) \cdot (-10\hat{i} - 2\hat{j} - 3\hat{k}) = -80 - 16 - 12 = -108$$

Substituting all the values in equation (1), we obtain

$$d = \left| \frac{-108}{12} \right| = 9$$

Therefore, the shortest distance between the two given lines is 9 units.

#### Question 10:

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the YZ-plane

Answer

It is known that the equation of the line passing through the points,  $(x_1, y_1, z_1)$  and  $(x_2,$

$$y_2, z_2), \text{ is } \frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

The line passing through the points, (5, 1, 6) and (3, 4, 1), is given by,

$$\begin{aligned} \frac{x-5}{3-5} &= \frac{y-1}{4-1} = \frac{z-6}{1-6} \\ \Rightarrow \frac{x-5}{-2} &= \frac{y-1}{3} = \frac{z-6}{-5} = k \text{ (say)} \\ \Rightarrow x &= 5 - 2k, y = 3k + 1, z = 6 - 5k \end{aligned}$$

Any point on the line is of the form  $(5 - 2k, 3k + 1, 6 - 5k)$ .

The equation of YZ-plane is  $x = 0$

Since the line passes through YZ-plane,

$$5 - 2k = 0$$

$$\Rightarrow k = \frac{5}{2}$$

$$\Rightarrow 3k + 1 = 3 \times \frac{5}{2} + 1 = \frac{17}{2}$$

$$6 - 5k = 6 - 5 \times \frac{5}{2} = \frac{-13}{2}$$

Therefore, the required point is  $\left(0, \frac{17}{2}, \frac{-13}{2}\right)$ .

### Question 11:

Find the coordinates of the point where the line through (5, 1, 6) and (3, 4, 1) crosses the ZX – plane.

Answer

It is known that the equation of the line passing through the points,  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ , is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

The line passing through the points, (5, 1, 6) and (3, 4, 1), is given by,

$$\frac{x - 5}{3 - 5} = \frac{y - 1}{4 - 1} = \frac{z - 6}{1 - 6}$$

$$\Rightarrow \frac{x - 5}{-2} = \frac{y - 1}{3} = \frac{z - 6}{-5} = k \text{ (say)}$$

$$\Rightarrow x = 5 - 2k, y = 3k + 1, z = 6 - 5k$$

Any point on the line is of the form  $(5 - 2k, 3k + 1, 6 - 5k)$ .

Since the line passes through ZX-plane,

$$3k + 1 = 0$$

$$\Rightarrow k = -\frac{1}{3}$$

$$\Rightarrow 5 - 2k = 5 - 2\left(-\frac{1}{3}\right) = \frac{17}{3}$$

$$6 - 5k = 6 - 5\left(-\frac{1}{3}\right) = \frac{23}{3}$$

Therefore, the required point is  $\left(\frac{17}{3}, 0, \frac{23}{3}\right)$ .

**Question 12:**

Find the coordinates of the point where the line through  $(3, -4, -5)$  and  $(2, -3, 1)$  crosses the plane  $2x + y + z = 7$ .

Answer

It is known that the equation of the line through the points,  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ , is

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Since the line passes through the points,  $(3, -4, -5)$  and  $(2, -3, 1)$ , its equation is given by,

$$\begin{aligned} \frac{x-3}{2-3} &= \frac{y+4}{-3+4} = \frac{z+5}{1+5} \\ \Rightarrow \frac{x-3}{-1} &= \frac{y+4}{1} = \frac{z+5}{6} = k \text{ (say)} \\ \Rightarrow x &= 3-k, y = k-4, z = 6k-5 \end{aligned}$$

Therefore, any point on the line is of the form  $(3 - k, k - 4, 6k - 5)$ .

This point lies on the plane,  $2x + y + z = 7$

$$\therefore 2(3 - k) + (k - 4) + (6k - 5) = 7$$

$$\Rightarrow 5k - 3 = 7$$

$$\Rightarrow k = 2$$

Hence, the coordinates of the required point are  $(3 - 2, 2 - 4, 6 \times 2 - 5)$  i.e.,  $(1, -2, 7)$ .

**Question 13:**

Find the equation of the plane passing through the point  $(-1, 3, 2)$  and perpendicular to each of the planes  $x + 2y + 3z = 5$  and  $3x + 3y + z = 0$ .

Answer

The equation of the plane passing through the point  $(-1, 3, 2)$  is

$$a(x + 1) + b(y - 3) + c(z - 2) = 0 \dots (1)$$

where,  $a, b, c$  are the direction ratios of normal to the plane.

It is known that two planes,  $a_1x + b_1y + c_1z + d_1 = 0$  and  $a_2x + b_2y + c_2z + d_2 = 0$ , are perpendicular, if  $a_1a_2 + b_1b_2 + c_1c_2 = 0$

Plane (1) is perpendicular to the plane,  $x + 2y + 3z = 5$

$$\therefore a \cdot 1 + b \cdot 2 + c \cdot 3 = 0$$

$$\Rightarrow a + 2b + 3c = 0 \quad \dots(2)$$

Also, plane (1) is perpendicular to the plane,  $3x + 3y + z = 0$

$$\therefore a \cdot 3 + b \cdot 3 + c \cdot 1 = 0$$

$$\Rightarrow 3a + 3b + c = 0 \quad \dots(3)$$

From equations (2) and (3), we obtain

$$\frac{a}{2 \times 1 - 3 \times 3} = \frac{b}{3 \times 3 - 1 \times 1} = \frac{c}{1 \times 3 - 2 \times 3}$$

$$\Rightarrow \frac{a}{-7} = \frac{b}{8} = \frac{c}{-3} = k \text{ (say)}$$

$$\Rightarrow a = -7k, b = 8k, c = -3k$$

Substituting the values of  $a$ ,  $b$ , and  $c$  in equation (1), we obtain

$$-7k(x+1) + 8k(y-3) - 3k(z-2) = 0$$

$$\Rightarrow (-7x-7) + (8y-24) - 3z+6 = 0$$

$$\Rightarrow -7x + 8y - 3z - 25 = 0$$

$$\Rightarrow 7x - 8y + 3z + 25 = 0$$

This is the required equation of the plane.

#### Question 14:

If the points  $(1, 1, p)$  and  $(-3, 0, 1)$  be equidistant from the plane

$$\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0, \text{ then find the value of } p.$$

Answer

The position vector through the point  $(1, 1, p)$  is  $\vec{a}_1 = \hat{i} + \hat{j} + p\hat{k}$

Similarly, the position vector through the point  $(-3, 0, 1)$  is

$$\vec{a}_2 = -4\hat{i} + \hat{k}$$



The equation of the given plane is  $\vec{r} \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13 = 0$

It is known that the perpendicular distance between a point whose position vector is

$$\vec{a} \text{ and the plane, } \vec{r} \cdot \vec{N} = d, \text{ is given by, } D = \frac{|\vec{a} \cdot \vec{N} - d|}{|\vec{N}|}$$

Here,  $\vec{N} = 3\hat{i} + 4\hat{j} - 12\hat{k}$  and  $d = -13$

Therefore, the distance between the point  $(1, 1, p)$  and the given plane is

$$\begin{aligned} D_1 &= \frac{|(\hat{i} + \hat{j} + p\hat{k}) \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13|}{|3\hat{i} + 4\hat{j} - 12\hat{k}|} \\ \Rightarrow D_1 &= \frac{|3 + 4 - 12p + 13|}{\sqrt{3^2 + 4^2 + (-12)^2}} \\ \Rightarrow D_1 &= \frac{|20 - 12p|}{13} \quad \dots(1) \end{aligned}$$

Similarly, the distance between the point  $(-3, 0, 1)$  and the given plane is

$$\begin{aligned} D_2 &= \frac{|(-3\hat{i} + \hat{k}) \cdot (3\hat{i} + 4\hat{j} - 12\hat{k}) + 13|}{|3\hat{i} + 4\hat{j} - 12\hat{k}|} \\ \Rightarrow D_2 &= \frac{|-9 - 12 + 13|}{\sqrt{3^2 + 4^2 + (-12)^2}} \\ \Rightarrow D_2 &= \frac{8}{13} \quad \dots(2) \end{aligned}$$

It is given that the distance between the required plane and the points,  $(1, 1, p)$  and  $(-3, 0, 1)$ , is equal.

$$\therefore D_1 = D_2$$

$$\Rightarrow \frac{|20 - 12p|}{13} = \frac{8}{13}$$

$$\Rightarrow 20 - 12p = 8 \text{ or } -(20 - 12p) = 8$$

$$\Rightarrow 12p = 12 \text{ or } 12p = 28$$

$$\Rightarrow p = 1 \text{ or } p = \frac{7}{3}$$

**Question 15:**

Find the equation of the plane passing through the line of intersection of the planes

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1 \quad \text{and} \quad \vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0 \quad \text{and parallel to x-axis.}$$

Answer

The given planes are

$$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 1$$

$$\Rightarrow \vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) - 1 = 0$$

$$\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4 = 0$$

The equation of any plane passing through the line of intersection of these planes is

$$\begin{aligned} & [\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) - 1] + \lambda [\vec{r} \cdot (2\hat{i} + 3\hat{j} - \hat{k}) + 4] = 0 \\ & \vec{r} \cdot [(2\lambda + 1)\hat{i} + (3\lambda + 1)\hat{j} + (1 - \lambda)\hat{k}] + (4\lambda + 1) = 0 \quad \dots(1) \end{aligned}$$

Its direction ratios are  $(2\lambda + 1)$ ,  $(3\lambda + 1)$ , and  $(1 - \lambda)$ .

The required plane is parallel to x-axis. Therefore, its normal is perpendicular to x-axis.

The direction ratios of x-axis are 1, 0, and 0.

$$\therefore 1 \cdot (2\lambda + 1) + 0(3\lambda + 1) + 0(1 - \lambda) = 0$$

$$\Rightarrow 2\lambda + 1 = 0$$

$$\Rightarrow \lambda = -\frac{1}{2}$$

Substituting  $\lambda = -\frac{1}{2}$  in equation (1), we obtain

$$\Rightarrow \vec{r} \cdot \left[ -\frac{1}{2}\hat{j} + \frac{3}{2}\hat{k} \right] + (-3) = 0$$

$$\Rightarrow \vec{r} \cdot (\hat{j} - 3\hat{k}) + 6 = 0$$

Therefore, its Cartesian equation is  $y - 3z + 6 = 0$

This is the equation of the required plane.

**Question 16:**

If O be the origin and the coordinates of P be  $(1, 2, -3)$ , then find the equation of the plane passing through P and perpendicular to OP.

Answer

The coordinates of the points, O and P, are  $(0, 0, 0)$  and  $(1, 2, -3)$  respectively.

Therefore, the direction ratios of OP are  $(1 - 0) = 1$ ,  $(2 - 0) = 2$ , and  $(-3 - 0) = -3$

It is known that the equation of the plane passing through the point  $(x_1, y_1, z_1)$  is

$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$  where, a, b, and c are the direction ratios of normal.

Here, the direction ratios of normal are 1, 2, and -3 and the point P is  $(1, 2, -3)$ .

Thus, the equation of the required plane is

$$1(x - 1) + 2(y - 2) - 3(z + 3) = 0$$

$$\Rightarrow x + 2y - 3z - 14 = 0$$

**Question 17:**

Find the equation of the plane which contains the line of intersection of the planes

$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0$  ,  $\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0$  and which is perpendicular to the plane

$\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$

Answer

The equations of the given planes are

$$\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4 = 0 \quad \dots(1)$$

$$\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5 = 0 \quad \dots(2)$$

The equation of the plane passing through the line intersection of the plane given in equation (1) and equation (2) is

$$[\vec{r} \cdot (\hat{i} + 2\hat{j} + 3\hat{k}) - 4] + \lambda [\vec{r} \cdot (2\hat{i} + \hat{j} - \hat{k}) + 5] = 0$$

$$\vec{r} \cdot [(2\lambda + 1)\hat{i} + (\lambda + 2)\hat{j} + (3 - \lambda)\hat{k}] + (5\lambda - 4) = 0 \quad \dots(3)$$

The plane in equation (3) is perpendicular to the plane,  $\vec{r} \cdot (5\hat{i} + 3\hat{j} - 6\hat{k}) + 8 = 0$

$$\therefore 5(2\lambda + 1) + 3(\lambda + 2) - 6(3 - \lambda) = 0$$

$$\Rightarrow 19\lambda - 7 = 0$$

$$\Rightarrow \lambda = \frac{7}{19}$$

Substituting  $\lambda = \frac{7}{19}$  in equation (3), we obtain

$$\Rightarrow \vec{r} \cdot \left[ \frac{33}{19}\hat{i} + \frac{45}{19}\hat{j} + \frac{50}{19}\hat{k} \right] - \frac{41}{19} = 0$$

$$\Rightarrow \vec{r} \cdot (33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 = 0 \quad \dots(4)$$

This is the vector equation of the required plane.

The Cartesian equation of this plane can be obtained by substituting  $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$  in equation (3).

$$(x\hat{i} + y\hat{j} + z\hat{k}) \cdot (33\hat{i} + 45\hat{j} + 50\hat{k}) - 41 = 0$$

$$\Rightarrow 33x + 45y + 50z - 41 = 0$$

### Question 18:

Find the distance of the point  $(-1, -5, -10)$  from the point of intersection of the line

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad \text{and the plane} \quad \vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$$

Answer

The equation of the given line is

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \quad \dots(1)$$

The equation of the given plane is

$$\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5 \quad \dots(2)$$

Substituting the value of  $\vec{r}$  from equation (1) in equation (2), we obtain

$$\begin{aligned} & \left[ 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k}) \right] \cdot (\hat{i} - \hat{j} + \hat{k}) = 5 \\ \Rightarrow & \left[ (3\lambda + 2)\hat{i} + (4\lambda - 1)\hat{j} + (2\lambda + 2)\hat{k} \right] \cdot (\hat{i} - \hat{j} + \hat{k}) = 5 \\ \Rightarrow & (3\lambda + 2) - (4\lambda - 1) + (2\lambda + 2) = 5 \\ \Rightarrow & \lambda = 0 \end{aligned}$$

Substituting this value in equation (1), we obtain the equation of the line as

$$\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$$

This means that the position vector of the point of intersection of the line and the plane

$$\text{is } \vec{r} = 2\hat{i} - \hat{j} + 2\hat{k}$$

This shows that the point of intersection of the given line and plane is given by the coordinates, (2, -1, 2). The point is (-1, -5, -10).

The distance  $d$  between the points, (2, -1, 2) and (-1, -5, -10), is

$$d = \sqrt{(-1-2)^2 + (-5+1)^2 + (-10-2)^2} = \sqrt{9+16+144} = \sqrt{169} = 13$$

### Question 19:

Find the vector equation of the line passing through (1, 2, 3) and parallel to the planes

$$\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5 \quad \text{and} \quad \vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6$$

Answer

Let the required line be parallel to vector  $\vec{b}$  given by,

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

The position vector of the point (1, 2, 3) is  $\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

The equation of line passing through (1, 2, 3) and parallel to  $\vec{b}$  is given by,

$$\begin{aligned} \vec{r} &= \vec{a} + \lambda\vec{b} \\ \Rightarrow \vec{r} &= (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda(b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \quad \dots(1) \end{aligned}$$

The equations of the given planes are

$$\vec{r} \cdot (\hat{i} - \hat{j} + 2\hat{k}) = 5 \quad \dots(2)$$

$$\vec{r} \cdot (3\hat{i} + \hat{j} + \hat{k}) = 6 \quad \dots(3)$$

The line in equation (1) and plane in equation (2) are parallel. Therefore, the normal to the plane of equation (2) and the given line are perpendicular.

$$\Rightarrow (\hat{i} - \hat{j} + 2\hat{k}) \cdot \lambda (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = 0$$

$$\Rightarrow \lambda (b_1 - b_2 + 2b_3) = 0$$

$$\Rightarrow b_1 - b_2 + 2b_3 = 0 \quad \dots(4)$$

$$\text{Similarly, } (3\hat{i} + \hat{j} + \hat{k}) \cdot \lambda (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) = 0$$

$$\Rightarrow \lambda (3b_1 + b_2 + b_3) = 0$$

$$\Rightarrow 3b_1 + b_2 + b_3 = 0 \quad \dots(5)$$

From equations (4) and (5), we obtain

$$\frac{b_1}{(-1) \times 1 - 1 \times 2} = \frac{b_2}{2 \times 3 - 1 \times 1} = \frac{b_3}{1 \times 1 - 3(-1)}$$

$$\Rightarrow \frac{b_1}{-3} = \frac{b_2}{5} = \frac{b_3}{4}$$

Therefore, the direction ratios of  $\vec{b}$  are  $-3$ ,  $5$ , and  $4$ .

$$\therefore \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k} = -3\hat{i} + 5\hat{j} + 4\hat{k}$$

Substituting the value of  $\vec{b}$  in equation (1), we obtain

$$\vec{r} = (\hat{i} + 2\hat{j} + 3\hat{k}) + \lambda (-3\hat{i} + 5\hat{j} + 4\hat{k})$$

This is the equation of the required line.

### Question 20:

Find the vector equation of the line passing through the point  $(1, 2, -4)$  and

perpendicular to the two lines:  $\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7}$  and  $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$

Answer

Let the required line be parallel to the vector  $\vec{b}$  given by,  $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

The position vector of the point  $(1, 2, -4)$  is  $\vec{a} = \hat{i} + 2\hat{j} - 4\hat{k}$

The equation of the line passing through  $(1, 2, -4)$  and parallel to vector  $\vec{b}$  is

$$\vec{r} = \vec{a} + \lambda \vec{b}$$

$$\Rightarrow \vec{r} (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda (b_1\hat{i} + b_2\hat{j} + b_3\hat{k}) \quad \dots(1)$$

The equations of the lines are

$$\frac{x-8}{3} = \frac{y+19}{-16} = \frac{z-10}{7} \quad \dots(2)$$

$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad \dots(3)$$

Line (1) and line (2) are perpendicular to each other.

$$\therefore 3b_1 - 16b_2 + 7b_3 = 0 \quad \dots(4)$$

Also, line (1) and line (3) are perpendicular to each other.

$$\therefore 3b_1 + 8b_2 - 5b_3 = 0 \quad \dots(5)$$

From equations (4) and (5), we obtain

$$\frac{b_1}{(-16)(-5) - 8 \times 7} = \frac{b_2}{7 \times 3 - 3(-5)} = \frac{b_3}{3 \times 8 - 3(-16)}$$

$$\Rightarrow \frac{b_1}{24} = \frac{b_2}{36} = \frac{b_3}{72}$$

$$\Rightarrow \frac{b_1}{2} = \frac{b_2}{3} = \frac{b_3}{6}$$

$\therefore$  Direction ratios of  $\vec{b}$  are 2, 3, and 6.

$$\therefore \vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$$

Substituting  $\vec{b} = 2\hat{i} + 3\hat{j} + 6\hat{k}$  in equation (1), we obtain

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda (2\hat{i} + 3\hat{j} + 6\hat{k})$$

This is the equation of the required line.

**Question 21:**

Prove that if a plane has the intercepts  $a, b, c$  and is at a distance of  $P$  units from the

origin, then  $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{1}{p^2}$

Answer

The equation of a plane having intercepts  $a, b, c$  with  $x, y,$  and  $z$  axes respectively is given by,

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \dots(1)$$

The distance ( $p$ ) of the plane from the origin is given by,

$$p = \frac{\left| \frac{0}{a} + \frac{0}{b} + \frac{0}{c} - 1 \right|}{\sqrt{\left(\frac{1}{a}\right)^2 + \left(\frac{1}{b}\right)^2 + \left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow p = \frac{1}{\sqrt{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}}$$

$$\Rightarrow p^2 = \frac{1}{\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$$

**Question 22:**

Distance between the two planes:  $2x + 3y + 4z = 4$  and  $4x + 6y + 8z = 12$  is

(A) 2 units (B) 4 units (C) 8 units

(D)  $\frac{2}{\sqrt{29}}$  units

Answer

The equations of the planes are

$$2x + 3y + 4z = 4 \quad \dots(1)$$



$$4x + 6y + 8z = 12$$

$$\Rightarrow 2x + 3y + 4z = 6 \quad \dots(2)$$

It can be seen that the given planes are parallel.

It is known that the distance between two parallel planes,  $ax + by + cz = d_1$  and  $ax + by + cz = d_2$ , is given by,

$$D = \left| \frac{d_2 - d_1}{\sqrt{a^2 + b^2 + c^2}} \right|$$

$$\Rightarrow D = \left| \frac{6 - 4}{\sqrt{(2)^2 + (3)^2 + (4)^2}} \right|$$

$$D = \frac{2}{\sqrt{29}}$$

Thus, the distance between the lines is  $\frac{2}{\sqrt{29}}$  units.

Hence, the correct answer is D.

### Question 23:

The planes:  $2x - y + 4z = 5$  and  $5x - 2.5y + 10z = 6$  are

(A) Perpendicular (B) Parallel (C) intersect  $y$ -axis

(C) passes through  $\left(0, 0, \frac{5}{4}\right)$

Answer

The equations of the planes are

$$2x - y + 4z = 5 \quad \dots (1)$$

$$5x - 2.5y + 10z = 6 \quad \dots (2)$$

It can be seen that,

$$\frac{a_1}{a_2} = \frac{2}{5}$$

$$\frac{b_1}{b_2} = \frac{-1}{-2.5} = \frac{2}{5}$$

$$\frac{c_1}{c_2} = \frac{4}{10} = \frac{2}{5}$$

$$\therefore \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

Therefore, the given planes are parallel.  
Hence, the correct answer is B.