

# DETERMINANTS

## 4.1 Overview

To every square matrix  $A = [a_{ij}]$  of order  $n$ , we can associate a number (real or complex) called determinant of the matrix  $A$ , written as  $\det A$ , where  $a_{ij}$  is the  $(i, j)$ th element of  $A$ .

If  $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ , then determinant of  $A$ , denoted by  $|A|$  (or  $\det A$ ), is given by

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

### Remarks

- (i) Only square matrices have determinants.
- (ii) For a matrix  $A$ ,  $|A|$  is read as determinant of  $A$  and not, as modulus of  $A$ .

### 4.1.1 Determinant of a matrix of order one

Let  $A = [a]$  be the matrix of order 1, then determinant of  $A$  is defined to be equal to  $a$ .

### 4.1.2 Determinant of a matrix of order two

Let  $A = [a_{ij}] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$  be a matrix of order 2. Then the determinant of  $A$  is defined as:  $\det(A) = |A| = ad - bc$ .

### 4.1.3 Determinant of a matrix of order three

The determinant of a matrix of order three can be determined by expressing it in terms of second order determinants which is known as expansion of a determinant along a row (or a column). There are six ways of expanding a determinant of order 3 corresponding to each of three rows ( $R_1$ ,  $R_2$  and  $R_3$ ) and three columns ( $C_1$ ,  $C_2$  and  $C_3$ ) and each way gives the same value.

Consider the determinant of a square matrix  $A = [a_{ij}]_{3 \times 3}$ , i.e.,

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Expanding  $|A|$  along  $C_1$ , we get

$$\begin{aligned} |A| &= a_{11} (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{21} (-1)^{2+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} + a_{31} (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} \\ &= a_{11}(a_{22} a_{33} - a_{23} a_{32}) - a_{21}(a_{12} a_{33} - a_{13} a_{32}) + a_{31}(a_{12} a_{23} - a_{13} a_{22}) \end{aligned}$$

**Remark** In general, if  $A = kB$ , where  $A$  and  $B$  are square matrices of order  $n$ , then  $|A| = k^n |B|$ ,  $n = 1, 2, 3$ .

#### 4.1.4 Properties of Determinants

For any square matrix  $A$ ,  $|A|$  satisfies the following properties.

- (i)  $|A'| = |A|$ , where  $A'$  = transpose of matrix  $A$ .
- (ii) If we interchange any two rows (or columns), then sign of the determinant changes.
- (iii) If any two rows or any two columns in a determinant are identical (or proportional), then the value of the determinant is zero.
- (iv) Multiplying a determinant by  $k$  means multiplying the elements of only one row (or one column) by  $k$ .
- (v) If we multiply each element of a row (or a column) of a determinant by constant  $k$ , then value of the determinant is multiplied by  $k$ .
- (vi) If elements of a row (or a column) in a determinant can be expressed as the sum of two or more elements, then the given determinant can be expressed as the sum of two or more determinants.

- (vii) If to each element of a row (or a column) of a determinant the equimultiples of corresponding elements of other rows (columns) are added, then value of determinant remains same.

### Notes:

- (i) If all the elements of a row (or column) are zeros, then the value of the determinant is zero.
- (ii) If value of determinant 'Δ' becomes zero by substituting  $x = \alpha$ , then  $x - \alpha$  is a factor of 'Δ'.
- (iii) If all the elements of a determinant above or below the main diagonal consists of zeros, then the value of the determinant is equal to the product of diagonal elements.

### 4.1.5 Area of a triangle

Area of a triangle with vertices  $(x_1, y_1)$ ,  $(x_2, y_2)$  and  $(x_3, y_3)$  is given by

$$\frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

### 4.1.6 Minors and co-factors

- (i) Minor of an element  $a_{ij}$  of the determinant of matrix A is the determinant obtained by deleting  $i^{\text{th}}$  row and  $j^{\text{th}}$  column, and it is denoted by  $M_{ij}$ .
- (ii) Co-factor of an element  $a_{ij}$  is given by  $A_{ij} = (-1)^{i+j} M_{ij}$ .
- (iii) Value of determinant of a matrix A is obtained by the sum of products of elements of a row (or a column) with corresponding co-factors. For example

$$|A| = a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}.$$

- (iv) If elements of a row (or column) are multiplied with co-factors of elements of any other row (or column), then their sum is zero. For example,

$$a_{11} A_{21} + a_{12} A_{22} + a_{13} A_{23} = 0.$$

### 4.1.7 Adjoint and inverse of a matrix

- (i) The adjoint of a square matrix  $A = [a_{ij}]_{n \times n}$  is defined as the transpose of the matrix

$[a_{ij}]_{n \times n}$ , where  $A_{ij}$  is the co-factor of the element  $a_{ij}$ . It is denoted by  $adj A$ .

$$\text{If } A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \text{ then } adj A = \begin{vmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{vmatrix}, \text{ where } A_{ij} \text{ is co-factor of } a_{ij}.$$

- (ii)  $A (adj A) = (adj A) A = |A| I$ , where  $A$  is square matrix of order  $n$ .
- (iii) A square matrix  $A$  is said to be singular or non-singular according as  $|A| = 0$  or  $|A| \neq 0$ , respectively.
- (iv) If  $A$  is a square matrix of order  $n$ , then  $|adj A| = |A|^{n-1}$ .
- (v) If  $A$  and  $B$  are non-singular matrices of the same order, then  $AB$  and  $BA$  are also nonsingular matrices of the same order.
- (vi) The determinant of the product of matrices is equal to product of their respective determinants, that is,  $|AB| = |A| |B|$ .
- (vii) If  $AB = BA = I$ , where  $A$  and  $B$  are square matrices, then  $B$  is called inverse of  $A$  and is written as  $B = A^{-1}$ . Also  $B^{-1} = (A^{-1})^{-1} = A$ .
- (viii) A square matrix  $A$  is invertible if and only if  $A$  is non-singular matrix.
- (ix) If  $A$  is an invertible matrix, then  $A^{-1} = \frac{1}{|A|} (adj A)$

#### 4.1.8 System of linear equations

- (i) Consider the equations:
 
$$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3, \end{aligned}$$

In matrix form, these equations can be written as  $AX = B$ , where

$$A = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, X = \begin{vmatrix} x \\ y \\ z \end{vmatrix} \text{ and } B = \begin{vmatrix} d_1 \\ d_2 \\ d_3 \end{vmatrix}$$

- (ii) Unique solution of equation  $AX = B$  is given by  $X = A^{-1}B$ , where  $|A| \neq 0$ .

- (iii) A system of equations is consistent or inconsistent according as its solution exists or not.
- (iv) For a square matrix  $A$  in matrix equation  $AX = B$
- If  $|A| \neq 0$ , then there exists unique solution.
  - If  $|A| = 0$  and  $(adj A) B \neq 0$ , then there exists no solution.
  - If  $|A| = 0$  and  $(adj A) B = 0$ , then system may or may not be consistent.

## 4.2 Solved Examples

### Short Answer (S.A.)

**Example 1** If  $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$ , then find  $x$ .

**Solution** We have  $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 5 \\ 8 & 3 \end{vmatrix}$ . This gives

$$2x^2 - 40 = 18 - 40 \quad \Rightarrow \quad x^2 = 9 \quad \Rightarrow \quad x = \pm 3.$$

**Example 2** If  $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ ,  $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$ , then prove that  $\Delta + \Delta_1 = 0$ .

**Solution** We have  $\Delta_1 = \begin{vmatrix} 1 & 1 & 1 \\ yz & zx & xy \\ x & y & z \end{vmatrix}$

Interchanging rows and columns, we get

$$\Delta_1 = \begin{vmatrix} 1 & yz & x \\ 1 & zx & y \\ 1 & xy & z \end{vmatrix} = \frac{1}{xyz} \begin{vmatrix} x & xyz & x^2 \\ y & xyz & y^2 \\ z & xyz & z^2 \end{vmatrix}$$

$$= \frac{xyz}{xyz} \begin{vmatrix} x & 1 & x^2 \\ y & 1 & y^2 \\ z & 1 & z^2 \end{vmatrix} \quad \text{Interchanging } C_1 \text{ and } C_2$$

$$= (-1) \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} -$$

$$\Rightarrow \Delta_1 + \Delta = 0$$

**Example 3** Without expanding, show that

$$\begin{vmatrix} \operatorname{cosec}^2 & \cot^2 & 1 \\ \cot^2 & \operatorname{cosec}^2 & 1 \\ 42 & 40 & 2 \end{vmatrix} = 0.$$

**Solution** Applying  $C_1 \rightarrow C_1 - C_2 - C_3$ , we have

$$\begin{vmatrix} \operatorname{cosec}^2 - \cot^2 - 1 & \cot^2 & 1 \\ \cot^2 - \operatorname{cosec}^2 & 1 & \operatorname{cosec}^2 \\ 0 & 40 & 2 \end{vmatrix} = \begin{vmatrix} 0 & \cot^2\theta & 1 \\ 0 & \operatorname{cosec}^2\theta & -1 \\ 0 & 40 & 2 \end{vmatrix} = 0$$

**Example 4** Show that  $\begin{vmatrix} x & p & q \\ p & x & q \\ q & q & x \end{vmatrix} = (x-p)(x^2 + px - 2q^2)$

**Solution** Applying  $C_1 \rightarrow C_1 - C_2$ , we have

$$\begin{vmatrix} x & p & p & q \\ p & x & x & q \\ 0 & q & q & x \end{vmatrix} \quad (x-p) \begin{vmatrix} 1 & p & q \\ 1 & x & q \\ 0 & q & x \end{vmatrix}$$

$$= (x-p) \begin{vmatrix} 0 & p+x & 2q \\ -1 & x & q \\ 0 & q & x \end{vmatrix} \quad \text{Applying } R_1 \rightarrow R_1 + R_2$$

Expanding along  $C_1$ , we have

$$(x-p)(px-x^2-2q^2) = (x-p)(x^2-px-2q^2)$$

**Example 5** If  $\begin{vmatrix} 0 & b & a & c & a \\ a & b & 0 & c & b \\ a & c & b & c & 0 \end{vmatrix}$ , then show that is equal to zero.

**Solution** Interchanging rows and columns, we get  $\begin{vmatrix} 0 & a & b & a & c \\ b & a & 0 & b & c \\ c & a & c & b & 0 \end{vmatrix}$

Taking ‘-1’ common from  $R_1, R_2$  and  $R_3$ , we get

$$(-1)^3 \begin{vmatrix} 0 & b & a & c & a \\ a & b & 0 & c & b \\ a & c & b & c & 0 \end{vmatrix} -$$

$$\Rightarrow 2 = 0 \quad \text{or} \quad = 0$$

**Example 6** Prove that  $(A^{-1})' = (A')^{-1}$ , where A is an invertible matrix.

**Solution** Since A is an invertible matrix, so it is non-singular.

We know that  $|A| = |A'|$ . But  $|A| \neq 0$ . So  $|A'| \neq 0$  i.e.  $A'$  is invertible matrix.

Now we know that  $AA^{-1} = A^{-1}A = I$ .

Taking transpose on both sides, we get  $(A^{-1})' A' = A' (A^{-1})' = (I)' = I$

Hence  $(A^{-1})'$  is inverse of  $A'$ , i.e.,  $(A')^{-1} = (A^{-1})'$

**Long Answer (L.A.)**

**Example 7** If  $x = -4$  is a root of  $\begin{vmatrix} x & 2 & 3 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix} = 0$ , then find the other two roots.

**Solution** Applying  $R_1 \rightarrow (R_1 + R_2 + R_3)$ , we get

$$\begin{vmatrix} x+4 & x+4 & x+4 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

Taking  $(x+4)$  common from  $R_1$ , we get

$$(x+4) \begin{vmatrix} 1 & 1 & 1 \\ 1 & x & 1 \\ 3 & 2 & x \end{vmatrix}$$

Applying  $C_2 \rightarrow C_2 - C_1$ ,  $C_3 \rightarrow C_3 - C_1$ , we get

$$(x+4) \begin{vmatrix} 1 & 0 & 0 \\ 1 & x-1 & 0 \\ 3 & 1 & x-3 \end{vmatrix}$$

Expanding along  $R_1$ ,

$\Delta = (x+4)[(x-1)(x-3) - 0]$ . Thus,  $\Delta = 0$  implies

$$x = -4, 1, 3$$

**Example 8** In a triangle ABC, if

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 \sin A & 1 \sin B & 1 \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix} = 0,$$

then prove that  $\Delta ABC$  is an isosceles triangle.

**Solution** Let  $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 \sin A & 1 \sin B & 1 \sin C \\ \sin A + \sin^2 A & \sin B + \sin^2 B & \sin C + \sin^2 C \end{vmatrix}$



$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 \sin A & 1 \sin B & 1 \sin C \\ \cos^2 A & \cos^2 B & \cos^2 C \end{vmatrix} \quad R_3 \rightarrow R_3 - R_2$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 \sin A & \sin B - \sin A & \sin C - \sin B \\ \cos^2 A & \cos^2 A - \cos^2 B & \cos^2 A - \cos^2 C \end{vmatrix} \cdot (C_3 \rightarrow C_3 - C_2 \text{ and } C_2 \rightarrow C_2 - C_1)$$

Expanding along  $R_1$ , we get

$$\begin{aligned} \Delta &= (\sin B - \sin A) (\sin^2 C - \sin^2 B) - (\sin C - \sin B) (\sin^2 B - \sin^2 A) \\ &= (\sin B - \sin A) (\sin C - \sin B) (\sin C + \sin B) - (\sin C - \sin B) (\sin C + \sin A) (\sin C - \sin A) = 0 \end{aligned}$$

$$\Rightarrow \text{either } \sin B - \sin A = 0 \text{ or } \sin C - \sin B = 0 \text{ or } \sin C - \sin A = 0$$

$$\Rightarrow A = B \text{ or } B = C \text{ or } C = A$$

i.e. triangle ABC is isosceles.

**Example 9** Show that if the determinant  $\begin{vmatrix} 3 & 2 & \sin 3 \\ 7 & 8 & \cos 2 \\ 11 & 14 & 2 \end{vmatrix} = 0$ , then  $\sin \theta = 0$  or  $\frac{1}{2}$ .

**Solution** Applying  $R_2 \rightarrow R_2 + 4R_1$  and  $R_3 \rightarrow R_3 + 7R_1$ , we get

$$\begin{vmatrix} 3 & 2 & \sin 3 \\ 5 & 0 & \cos 2 + 4\sin 3 \\ 10 & 0 & 2 + 7\sin 3 \end{vmatrix} = 0$$

$$\text{or } 2 [5 (2 + 7 \sin 3\theta) - 10 (\cos 2\theta + 4\sin 3\theta)] = 0$$

$$\text{or } 2 + 7\sin 3\theta - 2\cos 2\theta - 8\sin 3\theta = 0$$

$$\text{or } 2 - 2\cos 2\theta - \sin 3\theta = 0$$

$$\sin \theta (4\sin^2 \theta + 4\sin \theta - 3) = 0$$

or  $\sin\theta = 0$  or  $(2\sin\theta - 1) = 0$  or  $(2\sin\theta + 3) = 0$

or  $\sin\theta = 0$  or  $\sin\theta = \frac{1}{2}$  (Why ?).

### Objective Type Questions

Choose the correct answer from the given four options in each of the Example 10 and 11.

**Example 10** Let  $\begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix}$  and  $\begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix}$ , then

(A)  $\Delta_1 = -\Delta$

(B)  $\Delta \neq \Delta_1$

(C)  $\Delta - \Delta_1 = 0$

(D) None of these

**Solution** (C) is the correct answer since  $\begin{vmatrix} A & B & C \\ x & y & z \\ zy & zx & xy \end{vmatrix} = \begin{vmatrix} A & x & yz \\ B & y & zx \\ C & z & xy \end{vmatrix}$

$$= \frac{1}{xyz} \begin{vmatrix} Ax & x^2 & xyz \\ By & y^2 & xyz \\ Cz & z^2 & xyz \end{vmatrix} = \frac{xyz}{xyz} \begin{vmatrix} Ax & x^2 & 1 \\ By & y^2 & 1 \\ Cz & z^2 & 1 \end{vmatrix} = \Delta$$

**Example 11** If  $x, y \in \mathbf{R}$ , then the determinant  $\begin{vmatrix} \cos x & \sin x & 1 \\ \sin x & \cos x & 1 \\ \cos(x-y) & \sin(x-y) & 0 \end{vmatrix}$  lies

in the interval

(A)  $\sqrt{2}, \sqrt{2}$

(B)  $[-1, 1]$

(C)  $\sqrt{2}, 1$

(D)  $1, \sqrt{2}$ ,

**Solution** The correct choice is A. Indeed applying  $R_3 \rightarrow R_3 - \cos y R_1 + \sin y R_2$ , we get

$$\begin{vmatrix} \cos x & \sin x & 1 \\ \sin x & \cos x & 1 \\ 0 & 0 & \sin y \cos y \end{vmatrix}.$$

Expanding along  $R_3$ , we have

$$\begin{aligned} \Delta &= (\sin y - \cos y) (\cos^2 x + \sin^2 x) \\ &= (\sin y - \cos y) = \sqrt{2} \frac{1}{\sqrt{2}} \sin y - \frac{1}{\sqrt{2}} \cos y \\ &= \sqrt{2} \cos \frac{\pi}{4} \sin y - \sin \frac{\pi}{4} \cos y = \sqrt{2} \sin \left( y - \frac{\pi}{4} \right) \end{aligned}$$

Hence  $-\sqrt{2} \leq \Delta \leq \sqrt{2}$ .

Fill in the blanks in each of the Examples 12 to 14.

**Example 12** If A, B, C are the angles of a triangle, then

$$\begin{vmatrix} \sin^2 A & \cot A & 1 \\ \sin^2 B & \cot B & 1 \\ \sin^2 C & \cot C & 1 \end{vmatrix} \dots\dots\dots$$

**Solution** Answer is 0. Apply  $R_2 \rightarrow R_2 - R_1, R_3 \rightarrow R_3 - R_1$ .

**Example 13** The determinant  $\Delta = \begin{vmatrix} \sqrt{23} + \sqrt{3} & \sqrt{5} & \sqrt{5} \\ \sqrt{15} + \sqrt{46} & 5 & \sqrt{10} \\ 3 + \sqrt{115} & \sqrt{15} & 5 \end{vmatrix}$  is equal to .....

**Solution** Answer is 0. Taking  $\sqrt{5}$  common from  $C_2$  and  $C_3$  and applying  $C_1 \rightarrow C_3 - \sqrt{3} C_2$ , we get the desired result.

**Example 14** The value of the determinant

$$\begin{vmatrix} \sin^2 23 & \sin^2 67 & \cos 180 \\ \sin^2 67 & \sin^2 23 & \cos^2 180 \\ \cos 180 & \sin^2 23 & \sin^2 67 \end{vmatrix} \dots\dots\dots$$

**Solution**  $\Delta = 0$ . Apply  $C_1 \rightarrow C_1 + C_2 + C_3$ .

State whether the statements in the Examples 15 to 18 is **True** or **False**.

**Example 15** The determinant

$$\begin{vmatrix} \cos(x+y) & \sin(x+y) & \cos 2y \\ \sin x & \cos x & \sin y \\ \cos x & \sin x & \cos y \end{vmatrix}$$

is independent of  $x$  only.

**Solution** True. Apply  $R_1 \rightarrow R_1 + \sin y R_2 + \cos y R_3$ , and expand

**Example 16** The value of

$$\begin{vmatrix} 1 & 1 & 1 \\ {}^n C_1 & {}^{n+2} C_1 & {}^{n+4} C_1 \\ {}^n C_2 & {}^{n+2} C_2 & {}^{n+4} C_2 \end{vmatrix} \text{ is } 8.$$

**Solution** True

**Example 17** If  $A = \begin{pmatrix} x & 5 & 2 \\ 2 & y & 3 \\ 1 & 1 & z \end{pmatrix}$ ,  $xyz = 80$ ,  $3x + 2y + 10z = 20$ , then

$$A \text{ adj. } A = \begin{pmatrix} 81 & 0 & 0 \\ 0 & 81 & 0 \\ 0 & 0 & 81 \end{pmatrix}.$$

**Solution** : False.

**Example 18** If  $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 2 & x \\ 2 & 3 & 1 \end{pmatrix}$ ,  $A^{-1} = \begin{pmatrix} \frac{1}{2} & 4 & \frac{5}{2} \\ \frac{1}{2} & 3 & \frac{3}{2} \\ \frac{1}{2} & y & \frac{1}{2} \end{pmatrix}$

then  $x = 1$ ,  $y = -1$ .

**Solution** True

### 4.3 EXERCISE

#### Short Answer (S.A.)

Using the properties of determinants in Exercises 1 to 6, evaluate:

1.  $\begin{vmatrix} x^2 & x & 1 & x & 1 \\ x & 1 & x & 1 & \end{vmatrix}$

2.  $\begin{vmatrix} a & x & y & z \\ x & a & y & z \\ x & y & a & z \end{vmatrix}$

3.  $\begin{vmatrix} 0 & xy^2 & xz^2 \\ x^2y & 0 & yz^2 \\ x^2z & zy^2 & 0 \end{vmatrix}$

4.  $\begin{vmatrix} 3x & x & y & x & z \\ x & y & 3y & z & y \\ x & z & y & z & 3z \end{vmatrix}$

5.  $\begin{vmatrix} x & 4 & x & x \\ x & x & 4 & x \\ x & x & x & 4 \end{vmatrix}$

6.  $\begin{vmatrix} a & b & c & 2a & 2a \\ 2b & b & c & a & 2b \\ 2c & 2c & c & a & b \end{vmatrix}$

Using the properties of determinants in Exercises 7 to 9, prove that:

7.  $\begin{vmatrix} y^2z^2 & yz & y & z \\ z^2x^2 & zx & z & x \\ x^2y^2 & xy & x & y \end{vmatrix} = 0$

8.  $\begin{vmatrix} y & z & z & y \\ z & z & x & x \\ y & x & x & y \end{vmatrix} = 4xyz$

9. 
$$\begin{vmatrix} a^2 & 2a & 2a & 1 & 1 \\ 2a & 1 & a & 2 & 1 \\ 3 & & 3 & & 1 \end{vmatrix} (a-1)^3$$

10. If  $A + B + C = 0$ , then prove that 
$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} = 0$$

11. If the co-ordinates of the vertices of an equilateral triangle with sides of length

' $a$ ' are  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ , then 
$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}^2 = \frac{3a^4}{4}.$$

12. Find the value of  $\theta$  satisfying 
$$\begin{bmatrix} 1 & 1 & \sin 3\theta \\ -4 & 3 & \cos 2\theta \\ 7 & -7 & -2 \end{bmatrix} = 0.$$

13. If 
$$\begin{vmatrix} 4 & x & 4 & x & 4 & x \\ 4 & x & 4 & x & 4 & x \\ 4 & x & 4 & x & 4 & x \end{vmatrix} = 0,$$
 then find values of  $x$ .

14. If  $a_1, a_2, a_3, \dots, a_r$  are in G.P., then prove that the determinant

$$\begin{vmatrix} a_{r-1} & a_{r-5} & a_{r-9} \\ a_{r-7} & a_{r-11} & a_{r-15} \\ a_{r-11} & a_{r-17} & a_{r-21} \end{vmatrix}$$
 is independent of  $r$ .

15. Show that the points  $(a+5, a-4), (a-2, a+3)$  and  $(a, a)$  do not lie on a straight line for any value of  $a$ .

16. Show that the  $\triangle ABC$  is an isosceles triangle if the determinant

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 + \cos A & 1 + \cos B & 1 + \cos C \\ \cos^2 A + \cos A & \cos^2 B + \cos B & \cos^2 C + \cos C \end{vmatrix} = 0.$$

17. Find  $A^{-1}$  if  $A = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$  and show that  $A^{-1} = \frac{A^2 - 3I}{2}$ .

**Long Answer (L.A.)**

18. If  $A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & -1 & -2 \\ 0 & -1 & 1 \end{bmatrix}$ , find  $A^{-1}$ .

Using  $A^{-1}$ , solve the system of linear equations  $x - 2y = 10$ ,  $2x - y - z = 8$ ,  $-2y + z = 7$ .

19. Using matrix method, solve the system of equations  $3x + 2y - 2z = 3$ ,  $x + 2y + 3z = 6$ ,  $2x - y + z = 2$ .

20. Given  $A = \begin{vmatrix} 2 & 2 & 4 \\ 4 & 2 & 4 \\ 2 & 1 & 5 \end{vmatrix}$ ,  $B = \begin{vmatrix} 1 & 1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{vmatrix}$ , find  $BA$  and use this to solve the

system of equations  $y + 2z = 7$ ,  $x - y = 3$ ,  $2x + 3y + 4z = 17$ .

21. If  $a + b + c \neq 0$  and  $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$ , then prove that  $a = b = c$ .

22. Prove that  $\begin{vmatrix} bc & a^2 & ca & b^2 & ab & c^2 \\ ca & b^2 & ab & c^2 & bc & a^2 \\ ab & c^2 & bc & a^2 & ca & b^2 \end{vmatrix}$  is divisible by  $a + b + c$  and find the

quotient.

23. If  $x + y + z = 0$ , prove that 
$$\begin{vmatrix} xa & yb & zc \\ yc & za & xb \\ zb & xc & ya \end{vmatrix} = xyz \begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

### Objective Type Questions (M.C.Q.)

Choose the correct answer from given four options in each of the Exercises from 24 to 37.

24. If  $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & 2 \\ 7 & 3 \end{vmatrix}$ , then value of  $x$  is

- (A) 3 (B)  $\pm 3$   
(C)  $\pm 6$  (D) 6

25. The value of determinant 
$$\begin{vmatrix} a-b & b+c & a \\ b-a & c+a & b \\ c-a & a+b & c \end{vmatrix}$$

- (A)  $a^3 + b^3 + c^3$  (B)  $3bc$   
(C)  $a^3 + b^3 + c^3 - 3abc$  (D) none of these

26. The area of a triangle with vertices  $(-3, 0)$ ,  $(3, 0)$  and  $(0, k)$  is 9 sq. units. The value of  $k$  will be

- (A) 9 (B) 3  
(C)  $-9$  (D) 6

27. The determinant 
$$\begin{vmatrix} b^2 & ab & b & c & bc & ac \\ ab & a^2 & a & b & b^2 & ab \\ bc & ac & c & a & ab & a^2 \end{vmatrix}$$
 equals

- (A)  $abc(b-c)(c-a)(a-b)$  (B)  $(b-c)(c-a)(a-b)$   
(C)  $(a+b+c)(b-c)(c-a)(a-b)$  (D) None of these



28. The number of distinct real roots of  $\begin{vmatrix} \sin x & \cos x & \cos x \\ \cos x & \sin x & \cos x \\ \cos x & \cos x & \sin x \end{vmatrix} = 0$  in the interval

$$-\frac{\pi}{4} \leq x \leq \frac{\pi}{4} \text{ is}$$

- (A) 0 (B) 2  
(C) 1 (D) 3
29. If A, B and C are angles of a triangle, then the determinant

$$\begin{vmatrix} 1 & \cos C & \cos B \\ \cos C & 1 & \cos A \\ \cos B & \cos A & 1 \end{vmatrix} \text{ is equal to}$$

- (A) 0 (B) -1  
(C) 1 (D) None of these
30. Let  $f(t) = \begin{vmatrix} \cos t & t & 1 \\ 2\sin t & t & 2t \\ \sin t & t & t \end{vmatrix}$ , then  $\lim_{t \rightarrow 0} \frac{f(t)}{t^2}$  is equal to

- (A) 0 (B) -1  
(C) 2 (D) 3

31. The maximum value of  $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & \sin \theta \\ 1 & \cos \theta & 1 \end{vmatrix}$  is ( $\theta$  is real number)

- (A)  $\frac{1}{2}$  (B)  $\frac{\sqrt{3}}{2}$   
(C)  $\sqrt{2}$  (D)  $\frac{2\sqrt{3}}{4}$

32. If  $f(x) = \begin{vmatrix} 0 & x & a & x & b \\ x & a & 0 & x & c \\ x & b & x & c & 0 \end{vmatrix}$ , then

- (A)  $f(a) = 0$  (B)  $f(b) = 0$   
 (C)  $f(0) = 0$  (D)  $f(1) = 0$

33. If  $A = \begin{vmatrix} & 2 & & 3 \\ 0 & 2 & 5 & \\ & 1 & 1 & 3 \end{vmatrix}$ , then  $A^{-1}$  exists if

- (A)  $\lambda = 2$  (B)  $\lambda \neq 2$   
 (C)  $\lambda \neq -2$  (D) None of these

34. If A and B are invertible matrices, then which of the following is not correct?

- (A)  $\text{adj } A = |A| \cdot A^{-1}$  (B)  $\det(A)^{-1} = [\det(A)]^{-1}$   
 (C)  $(AB)^{-1} = B^{-1} A^{-1}$  (D)  $(A + B)^{-1} = B^{-1} + A^{-1}$

35. If  $x, y, z$  are all different from zero and  $\begin{vmatrix} 1 & x & 1 & 1 \\ 1 & 1 & y & 1 \\ 1 & 1 & 1 & z \end{vmatrix} = 0$ , then value of

$x^{-1} + y^{-1} + z^{-1}$  is

- (A)  $x y z$  (B)  $x^{-1} y^{-1} z^{-1}$   
 (C)  $-x - y - z$  (D)  $-1$

36. The value of the determinant  $\begin{vmatrix} x & x & y & x & 2y \\ x & 2y & x & x & y \\ x & y & x & 2y & x \end{vmatrix}$  is

- (A)  $9x^2(x + y)$  (B)  $9y^2(x + y)$   
 (C)  $3y^2(x + y)$  (D)  $7x^2(x + y)$

37. There are two values of  $a$  which makes determinant,  $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ 2 & a & 1 \\ 0 & 4 & 2a \end{vmatrix} = 86$ , then

sum of these number is

- (A) 4 (B) 5  
(C) -4 (D) 9

Fill in the blanks

38. If  $A$  is a matrix of order  $3 \times 3$ , then  $|3A| = \underline{\hspace{2cm}}$ .

39. If  $A$  is invertible matrix of order  $3 \times 3$ , then  $|A^{-1}| = \underline{\hspace{2cm}}$ .

40. If  $x, y, z \in \mathbb{R}$ , then the value of determinant  $\begin{vmatrix} 2^x & 2^{-x} & 2^x & 2^{-x} & 1 \\ 3^x & 3^{-x} & 3^x & 3^{-x} & 1 \\ 4^x & 4^{-x} & 4^x & 4^{-x} & 1 \end{vmatrix}$  is

equal to  $\underline{\hspace{2cm}}$ .

41. If  $\cos 2\theta = 0$ , then  $\begin{vmatrix} 0 & \cos \theta & \sin \theta \\ \cos \theta & \sin \theta & 0 \\ \sin \theta & 0 & \cos \theta \end{vmatrix}^2 = \underline{\hspace{2cm}}$ .

42. If  $A$  is a matrix of order  $3 \times 3$ , then  $(A^2)^{-1} = \underline{\hspace{2cm}}$ .

43. If  $A$  is a matrix of order  $3 \times 3$ , then number of minors in determinant of  $A$  are  $\underline{\hspace{2cm}}$ .

44. The sum of the products of elements of any row with the co-factors of corresponding elements is equal to  $\underline{\hspace{2cm}}$ .

45. If  $x = -9$  is a root of  $\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix} = 0$ , then other two roots are  $\underline{\hspace{2cm}}$ .

46.  $\begin{vmatrix} 0 & xyz & x-z \\ y-x & 0 & y-z \\ z-x & z-y & 0 \end{vmatrix} = \underline{\hspace{2cm}}$ .

47. If  $f(x) = \begin{vmatrix} (1+x)^{17} & (1+x)^{19} & (1+x)^{23} \\ (1+x)^{23} & (1+x)^{29} & (1+x)^{34} \\ (1+x)^{41} & (1+x)^{43} & (1+x)^{47} \end{vmatrix} = A + Bx + Cx^2 + \dots$ , then

A = \_\_\_\_\_.

State True or False for the statements of the following Exercises:

48.  $A^3^{-1} = A^{-1}{}^3$ , where A is a square matrix and  $|A| \neq 0$ .

49.  $(aA)^{-1} = \frac{1}{a}A^{-1}$ , where  $a$  is any real number and A is a square matrix.

50.  $|A^{-1}| \neq |A|^{-1}$ , where A is non-singular matrix.

51. If A and B are matrices of order 3 and  $|A| = 5$ ,  $|B| = 3$ , then  $|3AB| = 27 \times 5 \times 3 = 405$ .

52. If the value of a third order determinant is 12, then the value of the determinant formed by replacing each element by its co-factor will be 144.

53.  $\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix} = 0$ , where  $a, b, c$  are in A.P.

54.  $|\text{adj. } A| = |A|^2$ , where A is a square matrix of order two.

55. The determinant  $\begin{vmatrix} \sin A & \cos A & \sin A + \cos B \\ \sin B & \cos A & \sin B + \cos B \\ \sin C & \cos A & \sin C + \cos B \end{vmatrix}$  is equal to zero.

56. If the determinant  $\begin{vmatrix} x & a & p & u & l & f \\ y & b & q & v & m & g \\ z & c & r+w & n & h \end{vmatrix}$  splits into exactly K determinants of

order 3, each element of which contains only one term, then the value of K is 8.

57. Let  $\begin{vmatrix} a & p & x \\ b & q & y \\ c & r & z \end{vmatrix} = 16$ , then  $\Delta_1 = \begin{vmatrix} p+x & a+x & a+p \\ q+y & b+y & b+q \\ r+z & c+z & c+r \end{vmatrix} = 32$ .

58. The maximum value of  $\begin{vmatrix} 1 & 1 & 1 \\ 1 & (1 \sin \theta) & 1 \\ 1 & 1 & 1 \cos \theta \end{vmatrix}$  is  $\frac{1}{2}$ .

