Chapter 13

PROBABILITY

13.1 Overview

13.1.1 *Conditional Probability*

If E and F are two events associated with the same sample space of a random experiment, then the conditional probability of the event E under the condition that the event F has occurred, written as P (E | F), is given by

$$P(E \mid F) = \frac{P(E \cap F)}{P(F)}, \quad P(F) \neq 0$$

13.1.2 *Properties of Conditional Probability*

Let E and F be events associated with the sample space S of an experiment. Then:

(i) $P(S \mid F) = P(F \mid F) = 1$

(ii) $P[(A \cup B) \mid F] = P(A \mid F) + P(B \mid F) - P[(A \cap B \mid F)]$,

where A and B are any two events associated with S.

(iii) $P(E' \mid F) = 1 - P(E \mid F)$

13.1.3 *Multiplication Theorem on Probability*

Let E and F be two events associated with a sample space of an experiment. Then

$$P(E \cap F) = P(E)\, P(F \mid E),\ P(E) \neq 0$$

$$= P(F)\, P(E \mid F),\ P(F) \neq 0$$

If E, F and G are three events associated with a sample space, then

$$P(E \cap F \cap G) = P(E)\, P(F \mid E)\, P(G \mid E \cap F)$$

13.1.4 *Independent Events*

Let E and F be two events associated with a sample space S. If the probability of occurrence of one of them is not affected by the occurrence of the other, then we say that the two events are independent. Thus, two events E and F will be independent, if

(a) $P(F \mid E) = P(F)$, provided $P(E) \neq 0$

(b) $P(E \mid F) = P(E)$, provided $P(F) \neq 0$

Using the multiplication theorem on probability, we have

(c) $P(E \cap F) = P(E)\, P(F)$

Three events A, B and C are said to be mutually independent if all the following conditions hold:

$$P(A \cap B) = P(A)\, P(B)$$

$$P(A \cap C) = P(A)\, P(C)$$

$$P(B \cap C) = P(B)\, P(C)$$

and $$P(A \cap B \cap C) = P(A)\, P(B)\, P(C)$$

13.1.5 *Partition of a Sample Space*

A set of events $E_1, E_2, ..., E_n$ is said to represent a partition of a sample space S if

(a) $E_i \cap E_j = \phi,\ i \neq j;\ i, j = 1, 2, 3, ..., n$

(b) $E_i \cup E_2 \cup ... \cup E_n = S$, and

(c) Each $E_i \neq \phi$, i. e, $P(E_i) > 0$ for all $i = 1, 2, ..., n$

13.1.6 *Theorem of Total Probability*

Let $\{E_1, E, ..., E_n\}$ be a partition of the sample space S. Let A be any event associated with S, then

$$P(A) = \sum_{j=1}^{n} P(E_j)\, P(A \mid E_j)$$

13.1.7 *Bayes' Theorem*

If $E_1, E_2, ..., E_n$ are mutually exclusive and exhaustive events associated with a sample space, and A is any event of non zero probability, then

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{\sum_{i=1}^{n} P(E_i)P(A \mid E_i)}$$

13.1.8 *Random Variable and its Probability Distribution*

A random variable is a real valued function whose domain is the sample space of a random experiment.

The probability distribution of a random variable X is the system of numbers

X :	x_1		x_2	...	x_n
P (X) :	p_1		p_2	...	p_n

where $p_i > 0$, $i = 1, 2, ..., n$, $\sum_{i=1}^{n} p_i = 1$.

13.1.9 *Mean and Variance of a Random Variable*

Let X be a random variable assuming values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$, respectively such that $p_i \geq 0$, $\sum_{i=1}^{n} p_i = 1$. Mean of X, denoted by μ [or expected value of X denoted by E (X)] is defined as

$$\mu = E\,(X) = \sum_{i=1}^{n} x_i p_i$$

and variance, denoted by σ^2, is defined as

$$\sigma^2 = \sum_{i=1}^{n} (x_i - \mu)^2 p_i = \sum_{i=1}^{n} x_i^2 p_i - \mu^2$$

or equivalently

$$\sigma^2 = E(X - \mu)^2$$

Standard deviation of the random variable X is defined as

$$= \sqrt{\text{variance (X)}} = \sqrt{\sum_{i=1}^{n} (x_i - \mu)^2 p_i}$$

13.1.10 *Bernoulli Trials*

Trials of a random experiment are called Bernoulli trials, if they satisfy the following conditions:

(i) There should be a finite number of trials

(ii) The trials should be independent

(iii) Each trial has exactly two outcomes: success or failure

(iv) The probability of success (or failure) remains the same in each trial.

13.1.11 *Binomial Distribution*

A random variable X taking values 0, 1, 2, ..., n is said to have a binomial distribution with parameters n and p, if its probability distibution is given by

$$P(X = r) = {}^{n}c_{r}\, p^{r} q^{n-r},$$

where $q = 1 - p$ and $r = 0, 1, 2, \ldots, n$.

13.2 Solved Examples

Short Answer (S. A.)

Example 1 A and B are two candidates seeking admission in a college. The probability that A is selected is 0.7 and the probability that exactly one of them is selected is 0.6. Find the probability that B is selected.

Solution Let p be the probability that B gets selected.

P (Exactly one of A, B is selected) = 0.6 (given)

P (A is selected, B is not selected; B is selected, A is not selected) = 0.6

$P(A \cap B') + P(A' \cap B) = 0.6$

$$P(A)\,P(B') + P(A')\,P(B) = 0.6$$

$$(0.7)(1 - p) + (0.3)\,p = 0.6$$

$$p = 0.25$$

Thus the probability that B gets selected is 0.25.

Example 2 The probability of simultaneous occurrence of at least one of two events A and B is p. If the probability that exactly one of A, B occurs is q, then prove that $P(A') + P(B') = 2 - 2p + q$.

Solution Since P (exactly one of A, B occurs) = q (given), we get

$$P(A \cup B) - P(A \cap B) = q$$

$$\Rightarrow \quad p - P(A \cap B) = q$$

$$\Rightarrow \quad P(A \cap B) = p - q$$

$$\Rightarrow \quad 1 - P(A' \cup B') = p - q$$

$$\Rightarrow \quad P(A' \cup B') = 1 - p + q$$

$$\Rightarrow \quad P(A') + P(B') - P(A' \cap B') = 1 - p + q$$

$$\Rightarrow \quad P(A') + P(B') = (1 - p + q) + P(A' \cap B')$$

$$= (1 - p + q) + (1 - P(A \cup B))$$

$$= (1 - p + q) + (1 - p)$$

$$= 2 - 2p + q.$$

Example 3 10% of the bulbs produced in a factory are of red colour and 2% are red and defective. If one bulb is picked up at random, determine the probability of its being defective if it is red.

Solution Let A and B be the events that the bulb is red and defective, respectively.

$$P(A) = \frac{10}{100} = \frac{1}{10},$$

$$P(A \quad B) = \frac{2}{100} = \frac{1}{50}$$

$$P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{1}{50} \times \frac{10}{1} = \frac{1}{5}$$

Thus the probability of the picked up bulb of its being defective, if it is red, is $\frac{1}{5}$.

Example 4 Two dice are thrown together. Let A be the event 'getting 6 on the first die' and B be the event 'getting 2 on the second die'. Are the events A and B independent?

Solution: A = {(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

B = {(1, 2), (2, 2), (3, 2), (4, 2), (5, 2), (6, 2)}

$A \cap B = \{(6, 2)\}$

$$P(A) \quad \frac{6}{36} \quad \frac{1}{6}, \quad P(B) \quad \frac{1}{6}, \qquad P(A \quad B) \quad \frac{1}{36}$$

Events A and B will be independent if

$P(A \cap B) = P(A)\,P(B)$

i.e., $\text{LHS} = P \;\; A \quad B \quad \frac{1}{36}, \; \text{RHS} = P \;\; A \;\; P \;\; B \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{36}$

Hence, A and B are independent.

Example 5 A committee of 4 students is selected at random from a group consisting 8 boys and 4 girls. Given that there is at least one girl on the committee, calculate the probability that there are exactly 2 girls on the committee.

Solution Let A denote the event that at least one girl will be chosen, and B the event that exactly 2 girls will be chosen. We require P (B | A).

SinceA denotes the event that at least one girl will be chosen, A denotes that no girl is chosen, i.e., 4 boys are chosen. Then

$$P(A') = \frac{{}^{8}C_4}{{}^{12}C_4} = \frac{70}{495} = \frac{14}{99}$$

$$P(A) \quad 1-\frac{14}{99} \quad \frac{85}{99}$$

Now $\quad P(A \cap B) = P(2 \text{ boys and } 2 \text{ girls}) = \dfrac{{}^{8}C_{2}\,.\,{}^{4}C_{2}}{{}^{12}C_{4}}$

$$= \frac{6 \times 28}{495} = \frac{56}{165}$$

Thus $P(B \mid A) \quad = \dfrac{P(A \cap B)}{P(A)} = \dfrac{56}{165} \times \dfrac{99}{85} = \dfrac{168}{425}$

Example 6 Three machines E_1, E_2, E_3 in a certain factory produce 50%, 25% and 25%, respectively, of the total daily output of electric tubes. It is known that 4% of the tubes produced one each of machines E_1 and E_2 are defective, and that 5% of those produced on E_3 are defective. If one tube is picked up at random from a day's production, calculate the probability that it is defective.

Solution: Let D be the event that the picked up tube is defective

Let A_1 , A_2 and A_3 be the events that the tube is produced on machines E_1 , E_2 and E_3, respectively .

$$P(D) = P(A_1)\, P(D \mid A_1) + P(A_2)\, P(D \mid A_2) + P(A_3)\, P(D \mid A_3) \qquad (1)$$

$$P(A_1) = \frac{50}{100} = \frac{1}{2}, \; P(A_2) = \frac{1}{4}, \; P(A_3) = \frac{1}{4}$$

Also $\qquad P(D \mid A_1) = P(D \mid A_2) = \dfrac{4}{100} = \dfrac{1}{25}$

$$P(D \mid A_3) = \frac{5}{100} = \frac{1}{20}.$$

Putting these values in (1), we get

$$P(D) = \frac{1}{2} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{25} + \frac{1}{4} \times \frac{1}{20}$$

$$= \frac{1}{50} + \frac{1}{100} + \frac{1}{80} = \frac{17}{400} = .0425$$

Example 7 Find the probability that in 10 throws of a fair die a score which is a multiple of 3 will be obtained in at least 8 of the throws.

Solution Here success is a score which is a multiple of 3 i.e., 3 or 6.

Therefore, $p\ (3 \text{ or } 6) = \frac{2}{6} \quad \frac{1}{3}$

The probability of r successes in 10 throws is given by

$$P\ (r) = {}^{10}C_r \quad \frac{1}{3}^{\ r} \quad \frac{2}{3}^{\ 10-r}$$

Now $\quad$ P (at least 8 successes) = P (8) + P (9) + P (10)

$$= {}^{10}C_8 \left(\frac{1}{3}\right)^8 \left(\frac{2}{3}\right)^2 + {}^{10}C_9 \left(\frac{1}{3}\right)^9 \left(\frac{2}{3}\right)^1 + {}^{10}C_{10} \left(\frac{1}{3}\right)^{10}$$

$$= \frac{1}{3^{10}}\ [45 \times 4 + 10 \times 2 + 1] \ = \ \frac{201}{3^{10}}\ .$$

Example 8 A discrete random variable X has the following probability distribution:

X	1	2	3	4	5	6	7
P (X)	C	2C	2C	3C	C^2	$2C^2$	$7C^2 + C$

Find the value of C. Also find the mean of the distribution.

Solution Since $\Sigma\ p_i = 1$, we have

$C + 2C + 2C + 3C + C^2 + 2C^2 + 7C^2 + C = 1$

i.e., $\quad 10C^2 + 9C - 1 = 0$

i.e. $\quad (10C - 1)\ (C + 1) = 0$

$\Rightarrow \quad C = \frac{1}{10}, \qquad C = -1$

Therefore, the permissible value of $C = \frac{1}{10}$ (Why?)

$$\text{Mean} = \sum_{i=1}^{n} x_i\, p_i = \sum_{i=1}^{7} x_i\, p_i$$

$$= 1\times\frac{1}{10}+2\times\frac{2}{10}+3\times\frac{2}{10}+4\times\frac{3}{10}+5\left(\frac{1}{10}\right)^2+6\times 2\left(\frac{1}{10}\right)^2+7\left(7\left(\frac{1}{10}\right)^2+\frac{1}{10}\right)$$

$$= \frac{1}{10}+\frac{4}{10}+\frac{6}{10}+\frac{12}{10}+\frac{5}{100}+\frac{12}{100}+\frac{49}{100}+\frac{7}{10}$$

$= 3.66.$

Long Answer (L.A.)

Example 9 Four balls are to be drawn without replacement from a box containing 8 red and 4 white balls. If X denotes the number of red ball drawn, find the probability distribution of X.

Solution Since 4 balls have to be drawn, therefore, X can take the values 0, 1, 2, 3, 4.

P (X = 0) = P (no red ball) = P (4 white balls)

$$= \frac{^{4}C_{4}}{^{12}C_{4}} = \frac{1}{495}$$

P (X = 1) = P (1 red ball and 3 white balls)

$$= \frac{^{8}C_{1} \times {}^{4}C_{3}}{^{12}C_{4}} = \frac{32}{495}$$

P (X = 2) = P (2 red balls and 2 white balls)

$$= \frac{^{8}C_{2} \times {}^{4}C_{2}}{^{12}C_{4}} = \frac{168}{495}$$

P (X = 3) = P (3 red balls and 1 white ball)

$$= \frac{^{8}C_{3} \times {}^{4}C_{1}}{^{12}C_{4}} = \frac{224}{495}$$

$$P(X = 4) = P(4 \text{ red balls}) \quad \frac{{}^{8}C_{4}}{{}^{12}C_{4}} \quad \frac{70}{495}.$$

Thus the following is the required probability distribution of X

X	0	1	2	3	4
P (X)	$\frac{1}{495}$	$\frac{32}{495}$	$\frac{168}{495}$	$\frac{224}{495}$	$\frac{70}{495}$

Example 10 Determine variance and standard deviation of the number of heads in three tosses of a coin.

Solution Let X denote the number of heads tossed. So, X can take the values 0, 1, 2, 3. When a coin is tossed three times, we get

Sample space S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

$$P(X = 0) = P(\text{no head}) = P(\text{TTT}) = \frac{1}{8}$$

$$P(X = 1) = P(\text{one head}) = P(\text{HTT, THT, TTH}) = \frac{3}{8}$$

$$P(X = 2) = P(\text{two heads}) = P(\text{HHT, HTH, THH}) = \frac{3}{8}$$

$$P(X = 3) = P(\text{three heads}) = P(\text{HHH}) = \frac{1}{8}$$

Thus the probability distribution of X is:

X	0	1	2	3
P (X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Variance of X = $\sigma^2 = \Sigma x_i^2 p_i - \mu^2$, (1)
where μ is the mean of X given by

$$\mu = \Sigma x_i p_i = 0 \quad \frac{1}{8} \quad 1 \quad \frac{3}{8} \quad 2 \quad \frac{3}{8} \quad 3 \quad \frac{1}{8}$$

$$= \frac{3}{2} \qquad (2)$$

Now

$$\Sigma x_i^2 p_i = 0^2 \quad \frac{1}{8} \quad 1^2 \quad \frac{3}{8} \quad 2^2 \quad \frac{3}{8} \quad 3^2 \quad \frac{1}{8} \quad 3 \qquad (3)$$

From (1), (2) and (3), we get

$$\sigma^2 = 3 - \quad \frac{3}{2} \quad ^2 \quad \frac{3}{4}$$

$$\text{Standard deviation} \quad \sqrt{\quad ^2} \quad \sqrt{\frac{3}{4}} \quad \frac{\sqrt{3}}{2}.$$

Example 11 Refer to Example 6. Calculate the probability that the defective tube was produced on machine E_1.

Solution Now, we have to find P (A_1 / D).

$$P\,(A_1 / D) = \frac{P(A_1 \quad D)}{P(D)} \quad \frac{P(A_1)P(D/A_1)}{P(D)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{25}}{\frac{17}{400}} = \frac{8}{17}.$$

Example 12 A car manufacturing factory has two plants, X and Y. Plant X manufactures 70% of cars and plant Y manufactures 30%. 80% of the cars at plant X and 90% of the cars at plant Y are rated of standard quality. A car is chosen at random and is found to be of standard quality. What is the probability that it has come from plant X?

Solution Let E be the event that the car is of standard quality. Let B_1 and B_2 be the events that the car is manufactured in plants X and Y, respectively. Now

$$P\,(B_1) = \frac{70}{100} = \frac{7}{10}, \; P\,(B_2) = \frac{30}{100} = \frac{3}{10}$$

P $(E \mid B_1)$ = Probability that a standard quality car is manufactured in plant

$$= \frac{80}{100} = \frac{8}{10}$$

$$P(E \mid B_2) = \frac{90}{100} = \frac{9}{10}$$

$P(B_1 \mid E)$ = Probability that a standard quality car has come from plant X

$$= \frac{P(B_1) \times P(E \mid B_1)}{P(B_1) \,.\, P(E \mid B_1) + P(B_2) \,.\, P(E \mid B_2)}$$

$$= \frac{\frac{7}{10} \times \frac{8}{10}}{\frac{7}{10} \times \frac{8}{10} + \frac{3}{10} \times \frac{9}{10}} = \frac{56}{83}$$

Hence the required probability is $\frac{56}{83}$.

Objective Type Questions

Choose the correct answer from the given four options in each of the Examples 13 to 17.

Example 13 Let A and B be two events. If P (A) = 0.2, P (B) = 0.4, P (A∪B) = 0.6, then P (A | B) is equal to

(A) 0.8 (B) 0.5 (C) 0.3 (D) 0

Solution The correct answer is (D). From the given data P (A) + P (B) = P (A∪B).

This shows that P (A∩B) = 0. Thus P (A | B) = $\frac{P(A \quad B)}{P(B)}$ = 0.

Example 14 Let A and B be two events such that P (A) = 0.6, P (B) = 0.2, and P (A | B) = 0.5.

Then P (A′ | B′) equals

(A) $\frac{1}{10}$ (B) $\frac{3}{10}$ (C) $\frac{3}{8}$ (D) $\frac{6}{7}$

Solution The correct answer is (C). P (A∩B) = P (A | B) P (B)

= 0.5 × 0.2 = 0.1

$$P(A' \mid B') = \frac{P(A' \cap B')}{P(B')} = \frac{P[(A \cup B)']}{P(B')} = \frac{1 - P(A \cup B)}{1 - P(B)}$$

$$= \frac{1 - P(A) - P(B) + P(A \cap B)}{1 - 0.2} = \frac{3}{8}.$$

Example 15 If A and B are independent events such that $0 < P(A) < 1$ and $0 < P(B) < 1$, then which of the following is not correct?

(A) A and B are mutually exclusive
(B) A and B′ are independent
(C) A′ and B are independent
(D) A′ and B′ are independent

Solution The correct answer is (A).

Example 16 Let X be a discrete random variable. The probability distribution of X is given below:

X	30	10	– 10
P (X)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{1}{2}$

Then E (X) is equal to

(A) 6 (B) 4 (C) 3 (D) – 5

Solution The correct answer is (B).

$$E(X) = 30 \times \frac{1}{5} + 10 \times \frac{3}{10} - 10 \times \frac{1}{2} = 4.$$

Example 17 Let X be a discrete random variable assuming values $x_1, x_2, ..., x_n$ with probabilities $p_1, p_2, ..., p_n$, respectively. Then variance of X is given by

(A) $E(X^2)$ (B) $E(X^2) + E(X)$ (C) $E(X^2) - [E(X)]^2$

(D) $\sqrt{E(X^2) - [E(X)]^2}$

Solution The correct answer is (C).

Fill in the blanks in Examples 18 and 19

Example 18 If A and B are independent events such that P (A) = p, P (B) = $2p$ and

P (Exactly one of A, B) = $\frac{5}{9}$, then p = __________

Solution $p = \frac{1}{3}, \frac{5}{12}$ $\left[(1-p)(2p)+p(1-2p)=3p-4p^2=\frac{5}{9}\right]$

Example 19 If A and B′ are independent events then P (A′∪B) = 1 – ________

Solution P (A′∪B) = 1 – P (A∩B′) = 1 – P (A) P (B′)

(since A and B′ are independent).

State whether each of the statement in Examples 20 to 22 is **True** or **False**

Example 20 Let A and B be two independent events. Then P (A∩B) = P (A) + P (B)

Solution False, because P (A∩B) = P (A) . P(B) when events A and B are independent.

Example 21 Three events A, B and C are said to be independent if P (A∩B∩C) = P (A) P (B) P (C).

Solution False. Reason is that A, B, C will be independent if they are pairwise independent and P (A∩B∩C) = P (A) P (B) P (C).

Example 22 One of the condition of Bernoulli trials is that the trials are independent of each other.

Solution: True.

13.3 EXERCISE

Short Answer (S.A.)

1. For a loaded die, the probabilities of outcomes are given as under:
P(1) = P(2) = 0.2, P(3) = P(5) = P(6) = 0.1 and P(4) = 0.3.
The die is thrown two times. Let A and B be the events, 'same number each time', and 'a total score is 10 or more', respectively. Determine whether or not A and B are independent.

2. Refer to Exercise 1 above. If the die were fair, determine whether or not the events A and B are independent.

3. The probability that at least one of the two events A and B occurs is 0.6. If A and B occur simultaneously with probability 0.3, evaluate P($\overline{A}$) + P($\overline{B}$).

4. A bag contains 5 red marbles and 3 black marbles. Three marbles are drawn one by one without replacement. What is the probability that at least one of the three marbles drawn be black, if the first marble is red?

5. Two dice are thrown together and the total score is noted. The events E, F and G are 'a total of 4', 'a total of 9 or more', and 'a total divisible by 5', respectively. Calculate P(E), P(F) and P(G) and decide which pairs of events, if any, are independent.

6. Explain why the experiment of tossing a coin three times is said to have binomial distribution.

7. A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$ and $P(A \cap B) = \frac{1}{4}$.

Find :
(i) P(A|B) (ii) P(B|A) (iii) P(A'|B) (iv) P(A'|B')

8. Three events A, B and C have probabilities $\frac{2}{5}, \frac{1}{3}$ and $\frac{1}{2}$, respectively. Given that $P(A \cap C) = \frac{1}{5}$ and $P(B \cap C) = \frac{1}{4}$, find the values of $P(C \mid B)$ and $P(A' \cap C')$.

9. Let E_1 and E_2 be two independent events such that $p(E_1) = p_1$ and $P(E_2) = p_2$. Describe in words of the events whose probabilities are:
(i) $p_1\, p_2$ (ii) $(1-p_1)\, p_2$ (iii) $1-(1-p_1)(1-p_2)$ (iv) $p_1 + p_2 - 2p_1p_2$

10. A discrete random variable X has the probability distribution given as below:

X	0.5	1	1.5	2
P(X)	k	k^2	$2k^2$	k

(i) Find the value of k
(ii) Determine the mean of the distribution.

11. Prove that
(i) $P(A) = P(A \cap B) + P(A \cap \overline{B})$
(ii) $P(A \cup B) = P(A \cap B) + P(A \cap \overline{B}) + P(\overline{A} \cap B)$

12. If X is the number of tails in three tosses of a coin, determine the standard deviation of X.

13. In a dice game, a player pays a stake of Re1 for each throw of a die. She receives Rs 5 if the die shows a 3, Rs 2 if the die shows a 1 or 6, and nothing otherwise. What is the player's expected profit per throw over a long series of throws?

14. Three dice are thrown at the sametime. Find the probability of getting three two's, if it is known that the sum of the numbers on the dice was six.

15. Suppose 10,000 tickets are sold in a lottery each for Re 1. First prize is of Rs 3000 and the second prize is of Rs. 2000. There are three third prizes of Rs. 500 each. If you buy one ticket, what is your expectation.

16. A bag contains 4 white and 5 black balls. Another bag contains 9 white and 7 black balls. A ball is transferred from the first bag to the second and then a ball is drawn at random from the second bag. Find the probability that the ball drawn is white.

17. Bag I contains 3 black and 2 white balls, Bag II contains 2 black and 4 white balls. A bag and a ball is selected at random. Determine the probability of selecting a black ball.

18. A box has 5 blue and 4 red balls. One ball is drawn at random and not replaced. Its colour is also not noted. Then another ball is drawn at random. What is the probability of second ball being blue?

19. Four cards are successively drawn without replacement from a deck of 52 playing cards. What is the probability that all the four cards are kings?

20. A die is thrown 5 times. Find the probability that an odd number will come up exactly three times.

21. Ten coins are tossed. What is the probability of getting at least 8 heads?

22. The probability of a man hitting a target is 0.25. He shoots 7 times. What is the probability of his hitting at least twice?

23. A lot of 100 watches is known to have 10 defective watches. If 8 watches are selected (one by one with replacement) at random, what is the probability that there will be at least one defective watch?

24. Consider the probability distribution of a random variable X:

X	0	1	2	3	4
P(X)	0.1	0.25	0.3	0.2	0.15

Calculate (i) $V\left(\frac{X}{2}\right)$ (ii) Variance of X.

25. The probability distribution of a random variable X is given below:

X	0	1	2	3
P(X)	k	$\frac{k}{2}$	$\frac{k}{4}$	$\frac{k}{8}$

(i) Determine the value of k.

(ii) Determine $P(X \le 2)$ and $P(X > 2)$

(iii) Find $P(X \le 2) + P(X > 2)$.

26. For the following probability distribution determine standard deviation of the random variable X.

X	2	3	4
P(X)	0.2	0.5	0.3

27. A biased die is such that $P(4) = \frac{1}{10}$ and other scores being equally likely. The die is tossed twice. If X is the 'number of fours seen', find the variance of the random variable X.

28. A die is thrown three times. Let X be 'the number of twos seen'. Find the expectation of X.

29. Two biased dice are thrown together. For the first die $P(6) = \frac{1}{2}$, the other scores being equally likely while for the second die, $P(1) = \frac{2}{5}$ and the other scores are equally likely. Find the probability distribution of 'the number of ones seen'.

30. Two probability distributions of the discrete random variable X and Y are given below.

X	0	1	2	3
P(X)	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Y	0	1	2	3
P(Y)	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{10}$

Prove that $E(Y^2) = 2\,E(X)$.

31. A factory produces bulbs. The probability that any one bulb is defective is $\frac{1}{50}$ and they are packed in boxes of 10. From a single box, find the probability that

(i) none of the bulbs is defective

(ii) exactly two bulbs are defective

(iii) more than 8 bulbs work properly

32. Suppose you have two coins which appear identical in your pocket. You know that one is fair and one is 2-headed. If you take one out, toss it and get a head, what is the probability that it was a fair coin?

33. Suppose that 6% of the people with blood group O are left handed and 10% of those with other blood groups are left handed 30% of the people have blood group O. If a left handed person is selected at random, what is the probability that he/she will have blood group O?

34. Two natural numbers r, s are drawn one at a time, without replacement from the set S= $1, 2, 3, ..., n$. Find $P\left[r \le p | s \le p\right]$, where $p \in S$.

35. Find the probability distribution of the maximum of the two scores obtained when a die is thrown twice. Determine also the mean of the distribution.

36. The random variable X can take only the values 0, 1, 2. Given that $P(X = 0) = P(X = 1) = p$ and that $E(X^2) = E[X]$, find the value of p.

37. Find the variance of the distribution:

x	0	1	2	3	4	5
P(x)	$\frac{1}{6}$	$\frac{5}{18}$	$\frac{2}{9}$	$\frac{1}{6}$	$\frac{1}{9}$	$\frac{1}{18}$

38. A and B throw a pair of dice alternately. A wins the game if he gets a total of 6 and B wins if she gets a total of 7. It A starts the game, find the probability of winning the game by A in third throw of the pair of dice.

39. Two dice are tossed. Find whether the following two events A and B are independent:

$$A = \ (x, y) : x+y=11 \quad B = \ (x, y) : x \quad 5$$

where (x, y) denotes a typical sample point.

40. An urn contains m white and n black balls. A ball is drawn at random and is put back into the urn along with k additional balls of the same colour as that of the ball drawn. A ball is again drawn at random. Show that the probability of drawing a white ball now does not depend on k.

Long Answer (L.A.)

41. Three bags contain a number of red and white balls as follows:
Bag 1 : 3 red balls, Bag 2 : 2 red balls and 1 white ball
Bag 3 : 3 white balls.

The probability that bag i will be chosen and a ball is selected from it is $\frac{i}{6}$, $i = 1, 2, 3$. What is the probability that
(i) a red ball will be selected? (ii) a white ball is selected?

42. Refer to Question 41 above. If a white ball is selected, what is the probability that it came from

(i) Bag 2 (ii) Bag 3

43. A shopkeeper sells three types of flower seeds A_1, A_2 and A_3. They are sold as a mixture where the proportions are 4:4:2 respectively. The germination rates of the three types of seeds are 45%, 60% and 35%. Calculate the probability

(i) of a randomly chosen seed to germinate

(ii) that it will not germinate given that the seed is of type A_3,

(iii) that it is of the type A_2 given that a randomly chosen seed does not germinate.

44. A letter is known to have come either from TATA NAGAR or from CALCUTTA. On the envelope, just two consecutive letter TA are visible. What is the probability that the letter came from TATA NAGAR.

45. There are two bags, one of which contains 3 black and 4 white balls while the other contains 4 black and 3 white balls. A die is thrown. If it shows up 1 or 3, a ball is taken from the Ist bag; but it shows up any other number, a ball is chosen from the second bag. Find the probability of choosing a black ball.

46. There are three urns containing 2 white and 3 black balls, 3 white and 2 black balls, and 4 white and 1 black balls, respectively. There is an equal probability of each urn being chosen. A ball is drawn at random from the chosen urn and it is found to be white. Find the probability that the ball drawn was from the second urn.

47. By examining the chest X ray, the probability that TB is detected when a person is actually suffering is 0.99. The probability of an healthy person diagnosed to have TB is 0.001. In a certain city, 1 in 1000 people suffers from TB. A person is selected at random and is diagnosed to have TB. What is the probability that he actually has TB?

48. An item is manufactured by three machines A, B and C. Out of the total number of items manufactured during a specified period, 50% are manufactured on A, 30% on B and 20% on C. 2% of the items produced on A and 2% of items produced on B are defective, and 3% of these produced on C are defective. All the items are stored at one godown. One item is drawn at random and is found to be defective. What is the probability that it was manufactured on machine A?

49. Let X be a discrete random variable whose probability distribution is defined as follows:

$$P(X = x) = \begin{cases} k(x+1) & \text{for } x = 1,2,3,4 \\ 2kx & \text{for } x = 5,6,7 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Calculate

(i) the value of k (ii) E (X) (iii) Standard deviation of X.

50. The probability distribution of a discrete random variable X is given as under:

X	1	2	4	2A	3A	5A
P(X)	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{3}{25}$	$\frac{1}{10}$	$\frac{1}{25}$	$\frac{1}{25}$

Calculate :

(i) The value of A if E(X) = 2.94

(ii) Variance of X.

51. The probability distribution of a random variable x is given as under:

$$P(X = x) = \begin{cases} kx^2 & \text{for } x = 1, 2, 3 \\ 2kx & \text{for } x = 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant. Calculate

(i) E(X) (ii) E $(3X^2)$ (iii) P(X $\geq$ 4)

52. A bag contains $(2n + 1)$ coins. It is known that n of these coins have a head on both sides where as the rest of the coins are fair. A coin is picked up at random from the bag and is tossed. If the probability that the toss results in a head is $\frac{31}{42}$, determine the value of n.

53. Two cards are drawn successively without replacement from a well shuffled deck of cards. Find the mean and standard variation of the random variable X where X is the number of aces.

54. A die is tossed twice. A 'success' is getting an even number on a toss. Find the variance of the number of successes.

55. There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X.

Objective Type Questions

Choose the correct answer from the given four options in each of the exercises from 56 to 82.

56. If $P(A) = \frac{4}{5}$, and $P(A \cap B) = \frac{7}{10}$, then $P(B \mid A)$ is equal to

(A) $\frac{1}{10}$ (B) $\frac{1}{8}$ (C) $\frac{7}{8}$ (D) $\frac{17}{20}$

57. If $P(A \cap B) = \frac{7}{10}$ and $P(B) = \frac{17}{20}$, then $P(A \mid B)$ equals

(A) $\frac{14}{17}$ (B) $\frac{17}{20}$ (C) $\frac{7}{8}$ (D) $\frac{1}{8}$

58. If $P(A) = \frac{3}{10}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{3}{5}$, then $P(B \mid A) + P(A \mid B)$ equals

(A) $\frac{1}{4}$ (B) $\frac{1}{3}$ (C) $\frac{5}{12}$ (D) $\frac{7}{2}$

59. If $P(A) = \frac{2}{5}$, $P(B) = \frac{3}{10}$ and $P(A \cap B) = \frac{1}{5}$, then $P(A \mid B).P(B' \mid A')$ is equal to

(A) $\frac{5}{6}$ (B) $\frac{5}{7}$ (C) $\frac{25}{42}$ (D) 1

60. If A and B are two events such that $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{3}$, $P(A/B) = \frac{1}{4}$, then $P(A' \cap B')$ equals

(A) $\frac{1}{12}$ (B) $\frac{3}{4}$ (C) $\frac{1}{4}$ (D) $\frac{3}{16}$

61. If P(A) = 0.4, P(B) = 0.8 and P(B | A) = 0.6, then $P(A \cup B)$ is equal to

(A) 0.24 (B) 0.3 (C) 0.48 (D) 0.96

62. If A and B are two events and A ϕ, B ϕ, then

(A) P(A | B) = P(A).P(B) (B) $P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}$

(C) P(A | B).P(B | A)=1 (D) P(A | B) = P(A) | P(B)

63. A and B are events such that P(A) = 0.4, P(B) = 0.3 and $P(A \cup B)$ = 0.5. Then P (B A) equals

(A) $\frac{2}{3}$ (B) $\frac{1}{2}$ (C) $\frac{3}{10}$ (D) $\frac{1}{5}$

64. You are given that A and B are two events such that $P(B)=\frac{3}{5}$, $P(A \mid B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then P(A) equals

(A) $\frac{3}{10}$ (B) $\frac{1}{5}$ (C) $\frac{1}{2}$ (D) $\frac{3}{5}$

65. In Exercise 64 above, P(B | A) is equal to

(A) $\frac{1}{5}$ (B) $\frac{3}{10}$ (C) $\frac{1}{2}$ (D) $\frac{3}{5}$

66. If $P(B) = \frac{3}{5}$, $P(A \mid B) = \frac{1}{2}$ and $P(A \cup B) = \frac{4}{5}$, then $P(A \cup B)$ + $P(A \cup B)$ =

(A) $\frac{1}{5}$ (B) $\frac{4}{5}$ (C) $\frac{1}{2}$ (D) 1

67. Let $P(A) = \frac{7}{13}$, $P(B) = \frac{9}{13}$ and $P(A \cap B) = \frac{4}{13}$. Then $P(A \mid B)$ is equal to

(A) $\frac{6}{13}$ (B) $\frac{4}{13}$ (C) $\frac{4}{9}$ (D) $\frac{5}{9}$

68. If A and B are such events that $P(A) > 0$ and $P(B) \neq 1$, then $P(A \mid B)$ equals.

(A) $1 - P(A \mid B)$ (B) $1 - P(A \mid B)$

(C) $\frac{1 - P(A \cup B)}{P(B')}$ (D) $P(A) \mid P(B)$

69. If A and B are two independent events with $P(A) = \frac{3}{5}$ and $P(B) = \frac{4}{9}$, then $P(A \cap B)$ equals

(A) $\frac{4}{15}$ (B) $\frac{8}{45}$ (C) $\frac{1}{3}$ (D) $\frac{2}{9}$

70. If two events are independent, then

(A) they must be mutually exclusive

(B) the sum of their probabilities must be equal to 1

(C) (A) and (B) both are correct

(D) None of the above is correct

71. Let A and B be two events such that $P(A) = \frac{3}{8}$, $P(B) = \frac{5}{8}$ and $P(A \cup B) = \frac{3}{4}$. Then $P(A \mid B).P(A \mid B)$ is equal to

(A) $\frac{2}{5}$ (B) $\frac{3}{8}$ (C) $\frac{3}{20}$ (D) $\frac{6}{25}$

72. If the events A and B are independent, then $P(A \cap B)$ is equal to

(A) P (A) + P (B) (B) P(A) – P(B)

(C) P (A) . P(B) (D) P(A) | P(B)

73. Two events E and F are independent. If P(E) = 0.3, $P(E \cup F) = 0.5$, then P(E | F)–P(F | E) equals

(A) $\frac{2}{7}$ (B) $\frac{3}{35}$ (C) $\frac{1}{70}$ (D) $\frac{1}{7}$

74. A bag contains 5 red and 3 blue balls. If 3 balls are drawn at random without replacement the probability of getting exactly one red ball is

(A) $\frac{45}{196}$ (B) $\frac{135}{392}$ (C) $\frac{15}{56}$ (D) $\frac{15}{29}$

75. Refer to Question 74 above. The probability that exactly two of the three balls were red, the first ball being red, is

(A) $\frac{1}{3}$ (B) $\frac{4}{7}$ (C) $\frac{15}{28}$ (D) $\frac{5}{28}$

76. Three persons, A, B and C, fire at a target in turn, starting with A. Their probability of hitting the target are 0.4, 0.3 and 0.2 respectively. The probability of two hits is

(A) 0.024 (B) 0.188 (C) 0.336 (D) 0.452

77. Assume that in a family, each child is equally likely to be a boy or a girl. A family with three children is chosen at random. The probability that the eldest child is a girl given that the family has at least one girl is

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{2}{3}$ (D) $\frac{4}{7}$

78. A die is thrown and a card is selected at random from a deck of 52 playing cards. The probability of getting an even number on the die and a spade card is

(A) $\frac{1}{2}$ (B) $\frac{1}{4}$ (C) $\frac{1}{8}$ (D) $\frac{3}{4}$

79. A box contains 3 orange balls, 3 green balls and 2 blue balls. Three balls are drawn at random from the box without replacement. The probability of drawing 2 green balls and one blue ball is

(A) $\frac{3}{28}$ (B) $\frac{2}{21}$ (C) $\frac{1}{28}$ (D) $\frac{167}{168}$

80. A flashlight has 8 batteries out of which 3 are dead. If two batteries are selected without replacement and tested, the probability that both are dead is

(A) $\frac{33}{56}$ (B) $\frac{9}{64}$ (C) $\frac{1}{14}$ (D) $\frac{3}{28}$

81. Eight coins are tossed together. The probability of getting exactly 3 heads is

(A) $\frac{1}{256}$ (B) $\frac{7}{32}$ (C) $\frac{5}{32}$ (D) $\frac{3}{32}$

82. Two dice are thrown. If it is known that the sum of numbers on the dice was less than 6, the probability of getting a sum 3, is

(A) $\frac{1}{18}$ (B) $\frac{5}{18}$ (C) $\frac{1}{5}$ (D) $\frac{2}{5}$

83. Which one is not a requirement of a binomial distribution?

(A) There are 2 outcomes for each trial

(B) There is a fixed number of trials

(C) The outcomes must be dependent on each other

(D) The probability of success must be the same for all the trials

84. Two cards are drawn from a well shuffled deck of 52 playing cards with replacement. The probability, that both cards are queens, is

(A) $\frac{1}{13}\times\frac{1}{13}$ (B) $\frac{1}{13}+\frac{1}{13}$ (C) $\frac{1}{13}\times\frac{1}{17}$ (D) $\frac{1}{13}\times\frac{4}{51}$

85. The probability of guessing correctly at least 8 out of 10 answers on a true-false type examination is

(A) $\frac{7}{64}$ (B) $\frac{7}{128}$ (C) $\frac{45}{1024}$ (D) $\frac{7}{41}$

86. The probability that a person is not a swimmer is 0.3. The probability that out of 5 persons 4 are swimmers is

(A) ${}^5C_4\,(0.7)^4\,(0.3)$ (B) ${}^5C_1\,(0.7)\,(0.3)^4$

(C) ${}^5C_4\,(0.7)\,(0.3)^4$ (D) $(0.7)^4\,(0.3)$

87. The probability distribution of a discrete random variable X is given below:

X	2	3	4	5
P(X)	$\frac{5}{k}$	$\frac{7}{k}$	$\frac{9}{k}$	$\frac{11}{k}$

The value of k is

(A) 8 (B) 16 (C) 32 (D) 48

88. For the following probability distribution:

X	– 4	–3	–2	–1	0
P(X)	0.1	0.2	0.3	0.2	0.2

E(X) is equal to :

(A) 0 (B) –1 (C) –2 (D) –1.8

89. For the following probability distribution

X	1	2	3	4
P (X)	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

$E(X^2)$ is equal to

(A) 3 (B) 5 (C) 7 (D) 10

90. Suppose a random variable X follows the binomial distribution with parameters n and p, where $0 < p < 1$. If $P(x = r) / P(x = n–r)$ is independent of n and r, then p equals

(A) $\frac{1}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{5}$ (D) $\frac{1}{7}$

91. In a college, 30% students fail in physics, 25% fail in mathematics and 10% fail in both. One student is chosen at random. The probability that she fails in physics if she has failed in mathematics is

(A) $\frac{1}{10}$ (B) $\frac{2}{5}$ (C) $\frac{9}{20}$ (D) $\frac{1}{3}$

92. A and B are two students. Their chances of solving a problem correctly are $\frac{1}{3}$ and $\frac{1}{4}$, respectively. If the probability of their making a common error is, $\frac{1}{20}$ and they obtain the same answer, then the probability of their answer to be correct is

(A) $\frac{1}{12}$ (B) $\frac{1}{40}$ (C) $\frac{13}{120}$ (D) $\frac{10}{13}$

93. A box has 100 pens of which 10 are defective. What is the probability that out of a sample of 5 pens drawn one by one with replacement at most one is defective?

(A) $\left(\frac{9}{10}\right)^5$ (B) $\frac{1}{2}\left(\frac{9}{10}\right)^4$ (C) $\frac{1}{2}\left(\frac{9}{10}\right)^5$ (D) $\left(\frac{9}{10}\right)^5 + \frac{1}{2}\left(\frac{9}{10}\right)^4$

State True or False for the statements in each of the Exercises 94 to 103.

94. Let P(A) > 0 and P(B) > 0. Then A and B can be both mutually exclusive and independent.

95. If A and B are independent events, then A and B are also independent.

96. If A and B are mutually exclusive events, then they will be independent also.

97. Two independent events are always mutually exclusive.

98. If A and B are two independent events then P(A and B) = P(A).P(B).

99. Another name for the mean of a probability distribution is expected value.

100. If A and B′ are independent events, then $P(A' \cup B) = 1 - P(A)\,P(B')$

101. If A and B are independent, then

P (exactly one of A, B occurs) = $P(A)P(B\ ) + P\ B\ \ P\ A$

102. If A and B are two events such that P(A) > 0 and P(A) + P(B) >1, then

$$P(B \mid A) \geq 1 - \frac{P(B')}{P(A)}$$

103. If A, B and C are three independent events such that P(A) = P(B) = P(C) = p, then

P (At least two of A, B, C occur) = $3p^2 - 2p^3$

Fill in the blanks in each of the following questions:

104. If A and B are two events such that

$$P(A \mid B) = p,\ P(A) = p,\ P(B) = \frac{1}{3}$$

and $P(A \cup B) = \frac{5}{9}$, then p = _____

105. If A and B are such that

$$P(A' \cup B') = \frac{2}{3} \text{ and } P(A \cup B) = \frac{5}{9},$$

then P(A′) + P(B′) = ...

106. If X follows binomial distribution with parameters n = 5, p and P (X = 2) = 9, P (X = 3), then p = ____________

107. Let X be a random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$, respectively. Then var (X) = _________

108. Let A and B be two events. If P(A | B) = P(A), then A is ____________ of B.