Chapter 10

STRAIGHT LINES

❖ *Geometry, as a logical system, is a means and even the most powerful means to make children feel the strength of the human spirit that is of their own spirit.* – *H. FREUDENTHAL* ❖

10.1 Introduction

René Descartes (1596 -1650)

We are familiar with two-dimensional *coordinate geometry* from earlier classes. Mainly, it is a combination of *algebra* and *geometry*. A systematic study of geometry by the use of algebra was first carried out by celebrated French philosopher and mathematician René Descartes, in his book 'La Géométry, published in 1637. This book introduced the notion of the equation of a curve and related analytical methods into the study of geometry. The resulting combination of analysis and geometry is referred now as *analytical geometry*. In the earlier classes, we initiated the study of coordinate geometry, where we studied about coordinate axes, coordinate plane, plotting of points in a plane, distance between two points, section formule, etc. All these concepts are the basics of coordinate geometry.

Let us have a brief recall of coordinate geometry done in earlier classes. To recapitulate, the location of the points (6, – 4) and (3, 0) in the XY-plane is shown in Fig 10.1.

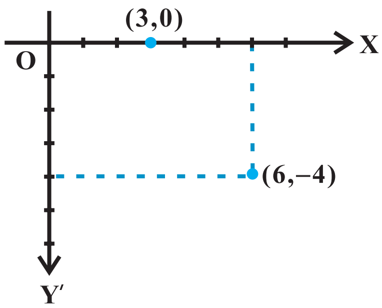

Fig 10.1

We may note that the point (6, – 4) is at 6 units distance from the *y*-axis measured along the positive *x*-axis and at 4 units distance from the *x*-axis measured along the negative *y*-axis. Similarly, the point (3, 0) is at 3 units distance from the *y*-axis measured along the positive *x*-axis and has zero distance from the *x*-axis.

We also studied there following important formulae:

I. Distance between the points P (x_1, y_1) and Q (x_2, y_2) is

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

For example, distance between the points (6, – 4) and (3, 0) is

$$\sqrt{(3-6)^2 + (0+4)^2} = \sqrt{9+16} = 5 \text{ units.}$$

II. The coordinates of a point dividing the line segment joining the points (x_1, y_1) and (x_2, y_2) internally, in the ratio $m: n$ are $\left(\dfrac{m x_2 + n x_1}{m+n}, \dfrac{m y_2 + n y_1}{m+n}\right)$.

For example, the coordinates of the point which divides the line segment joining A (1, –3) and B (–3, 9) internally, in the ratio 1: 3 are given by $x = \dfrac{1.(-3)+3.1}{1+3} = 0$

and $y = \dfrac{1.9+3.(-3)}{1+3} = 0.$

III. In particular, if $m = n$, the coordinates of the mid-point of the line segment joining the points (x_1, y_1) and (x_2, y_2) are $\left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$.

IV. Area of the triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\left| x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right| .$$

For example, the area of the triangle, whose vertices are (4, 4), (3, – 2) and (– 3, 16) is

$$\frac{1}{2}\left|4(-2-16)+3(16-4)+(-3)(4+2)\right| = \frac{|-54|}{2} = 27.$$

Remark If the area of the triangle ABC is zero, then three points A, B and C lie on a line, i.e., they are collinear.

In the this Chapter, we shall continue the study of coordinate geometry to study properties of the simplest geometric figure – *straight line.* Despite its simplicity, the line is a vital concept of geometry and enters into our daily experiences in numerous interesting and useful ways. Main focus is on representing the line algebraically, for which *slope* is most essential.

10.2 Slope of a Line

A line in a coordinate plane forms two angles with the x-axis, which are supplementary.

The angle (say) θ made by the line *l* with positive direction of *x*-axis and measured anti clockwise is called the *inclination of the line*. Obviously $0° \leq \theta \leq 180°$ (Fig 10.2).

We observe that lines parallel to *x*-axis, or coinciding with *x*-axis, have inclination of 0°. The inclination of a vertical line (parallel to or coinciding with *y*-axis) is 90°.

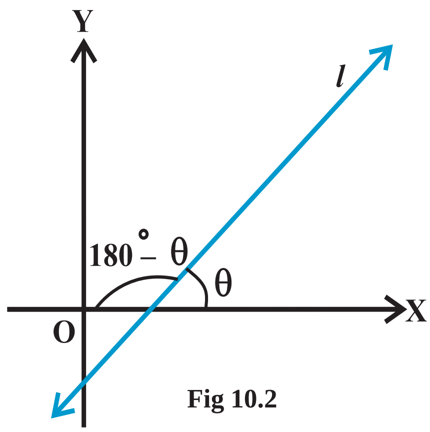

Fig 10.2

Definition 1 If θ is the inclination of a line *l*, then tan θ is called the *slope* or *gradient* of the line *l*.

The slope of a line whose inclination is 90° is not defined.

The slope of a line is denoted by *m*.

Thus, $m = \tan \theta, \theta \neq 90°$

It may be observed that the slope of *x*-axis is zero and slope of *y*-axis is not defined.

10.2.1 *Slope of a line when coordinates of any two points on the line are given*

We know that a line is completely determined when we are given two points on it. Hence, we proceed to find the slope of a line in terms of the coordinates of two points on the line.

Let $P(x_1, y_1)$ and $Q(x_2, y_2)$ be two points on non-vertical line *l* whose inclination is θ. Obviously, $x_1 \neq x_2$, otherwise the line will become perpendicular to *x*-axis and its slope will not be defined. The inclination of the line *l* may be acute or obtuse. Let us take these two cases.

Draw perpendicular QR to *x*-axis and PM perpendicular to RQ as shown in Figs. 10.3 (i) and (ii).

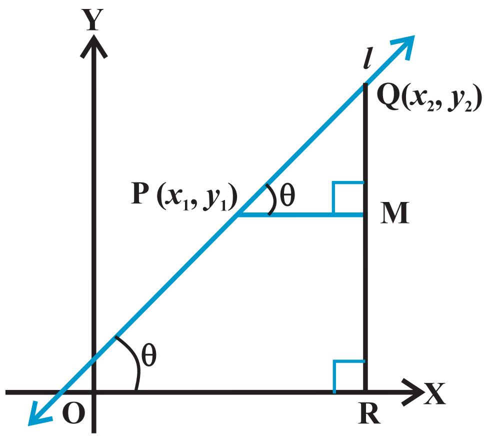

Fig 10. 3 (i)

Case 1 When angle θ is acute:

In Fig 10.3 (i), $\angle MPQ = \theta$. ... (1)

Therefore, slope of line $l = m = \tan \theta$.

But in ΔMPQ, we have $\tan\theta = \dfrac{MQ}{MP} = \dfrac{y_2 - y_1}{x_2 - x_1}$. ... (2)

From equations (1) and (2), we have

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Case II When angle θ is obtuse:
In Fig 10.3 (ii), we have
$\angle MPQ = 180° - \theta$.

Therefore, $\theta = 180° - \angle MPQ$.

Now, slope of the line l

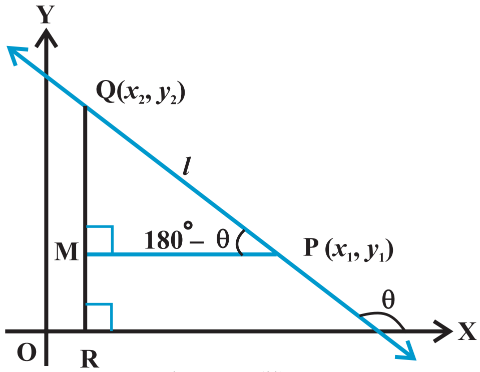

Fig 10. 3 (ii)

$$m = \tan\theta$$
$$= \tan(180° - \angle MPQ) = -\tan \angle MPQ$$
$$= -\frac{MQ}{MP} = -\frac{y_2 - y_1}{x_1 - x_2} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Consequently, we see that in both the cases the slope m of the line through the points (x_1, y_1) and (x_2, y_2) is given by $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

10.2.2 ***Conditions for parallelism and perpendicularity of lines in terms of their slopes*** In a coordinate plane, suppose that non-vertical *lines* l_1 and l_2 have slopes m_1 *and* m_2, respectively. Let their inclinations be α and β, respectively.

If the line l_1 is parallel to l_2 (Fig 10.4), then their inclinations are equal, i.e.,

$$\alpha = \beta, \text{ and hence, } \tan\alpha = \tan\beta$$

Therefore $m_1 = m_2$, i.e., their slopes are equal.

Conversely, if the slope of two lines l_1 and l_2 is same, i.e.,

$$m_1 = m_2.$$

Then $\tan\alpha = \tan\beta$.

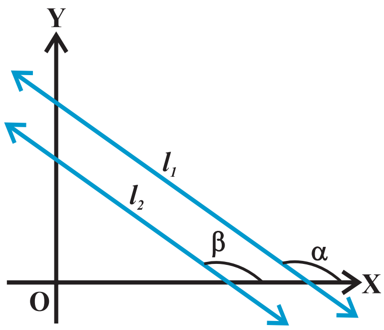

Fig 10. 4

By the property of tangent function (between 0° and 180°), $\alpha = \beta$.

Therefore, the lines are parallel.

Hence, two non vertical lines l_1 and l_2 are parallel if and only if their slopes are equal.

If the lines l_1 and l_2 are perpendicular (Fig 10.5), then $\beta = \alpha + 90°$.

Therefore, $\tan \beta = \tan (\alpha + 90°)$

$$= -\cot \alpha = -\frac{1}{\tan \alpha}$$

i.e., $m_2 = -\dfrac{1}{m_1}$ or $m_1 \, m_2 = -1$

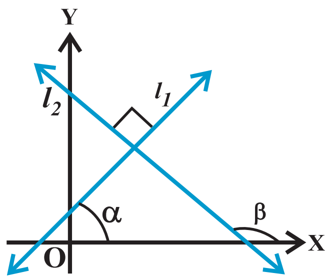

Fig 10. 5

Conversely, if $m_1 \, m_2 = -1$, i.e., $\tan \alpha \tan \beta = -1$.

Then $\tan \alpha = -\cot \beta = \tan (\beta + 90°)$ or $\tan (\beta - 90°)$

Therefore, α and β differ by 90°.

Thus, lines l_1 and l_2 are perpendicular to each other.

Hence, *two non-vertical lines are perpendicular to each other if and only if their slopes are negative reciprocals of each other,*

i.e., $m_2 = -\dfrac{1}{m_1}$ or, $m_1 \, m_2 = -1$.

Let us consider the following example.

Example 1 Find the slope of the lines:

(a) Passing through the points (3, – 2) and (–1, 4),

(b) Passing through the points (3, – 2) and (7, – 2),

(c) Passing through the points (3, – 2) and (3, 4),

(d) Making inclination of 60° with the positive direction of *x*-axis.

Solution (a) The slope of the line through (3, – 2) and (– 1, 4) is

$$m = \frac{4-(-2)}{-1-3} = \frac{6}{-4} = -\frac{3}{2}.$$

(b) The slope of the line through the points (3, – 2) and (7, – 2) is

$$m = \frac{-2-(-2)}{7-3} = \frac{0}{4} = 0.$$

(c) The slope of the line through the points (3, – 2) and (3, 4) is

$$m = \frac{4-(-2)}{3-3} = \frac{6}{0}, \text{ which is not defined.}$$

(d) Here inclination of the line $\alpha = 60°$. Therefore, slope of the line is

$$m = \tan 60° = \sqrt{3}.$$

10.2.3 ***Angle between two lines*** When we think about more than one line in a plane, then we find that these lines are either intersecting or parallel. Here we will discuss the angle between two lines in terms of their slopes.

Let L_1 and L_2 be two non-vertical lines with slopes m_1 and m_2, respectively. If α_1 and α_2 are the inclinations of lines L_1 and L_2, respectively. Then

$$m_1 = \tan\alpha_1 \text{ and } m_2 = \tan\alpha_2.$$

We know that when two lines intersect each other, they make two pairs of vertically opposite angles such that sum of any two adjacent angles is 180°. Let θ and ϕ be the adjacent angles between the lines L_1 and L_2 (Fig10.6). Then

$$\theta = \alpha_2 - \alpha_1 \text{ and } \alpha_1, \alpha_2 \neq 90°.$$

Therefore $\tan\theta = \tan(\alpha_2 - \alpha_1) = \dfrac{\tan\alpha_2 - \tan\alpha_1}{1+\tan\alpha_1 \tan\alpha_2} = \dfrac{m_2 - m_1}{1+m_1 m_2}$ (as $1 + m_1 m_2 \neq 0$)

and $\phi = 180° - \theta$ so that

$$\tan\phi = \tan(180° - \theta) = -\tan\theta = -\frac{m_2 - m_1}{1+m_1 m_2}, \text{ as } 1 + m_1 m_2 \neq 0$$

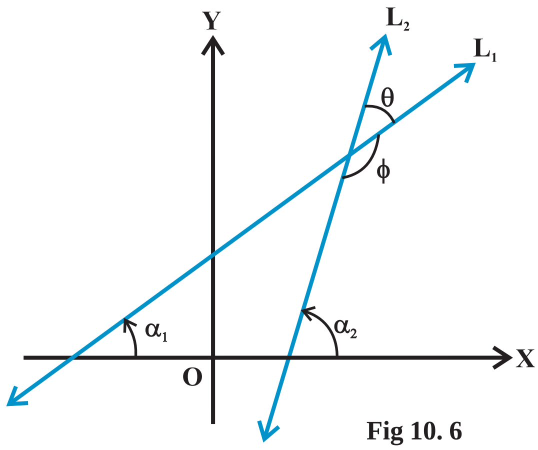

Fig 10. 6

Now, there arise two cases:

Case I If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is positive, then tan θ will be positive and tan ϕ will be negative, which means θ will be acute and ϕ will be obtuse.

Case II If $\frac{m_2 - m_1}{1 + m_1 m_2}$ is negative, then tan θ will be negative and tan ϕ will be positive, which means that θ will be obtuse and ϕ will be acute.

Thus, the acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2, respectively, is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right|, \text{ as } \quad 1 + m_1 m_2 \neq 0 \qquad \dots (1)$$

The obtuse angle (say ϕ) can be found by using $\phi = 180^0 - \theta$.

Example 2 If the angle between two lines is $\frac{\pi}{4}$ and slope of one of the lines is $\frac{1}{2}$, find the slope of the other line.

Solution We know that the acute angle θ between two lines with slopes m_1 and m_2 is given by

$$\tan\theta = \left|\frac{m_2 - m_1}{1 + m_1 m_2}\right| \qquad \dots (1)$$

Let $m_1 = \frac{1}{2}$, $m_2 = m$ and $\theta = \frac{\pi}{4}$.

Now, putting these values in (1), we get

$$\tan\frac{\pi}{4} = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right| \quad \text{or} \quad 1 = \left|\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m}\right|,$$

which gives
$$\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = 1 \quad \text{or} \quad -\frac{m - \frac{1}{2}}{1 + \frac{1}{2}m} = -1.$$

Therefore $\quad m = 3 \quad \text{or} \quad m = -\frac{1}{3}.$

Hence, slope of the other line is 3 or $-\frac{1}{3}$. Fig 10.7 explains the reason of two answers.

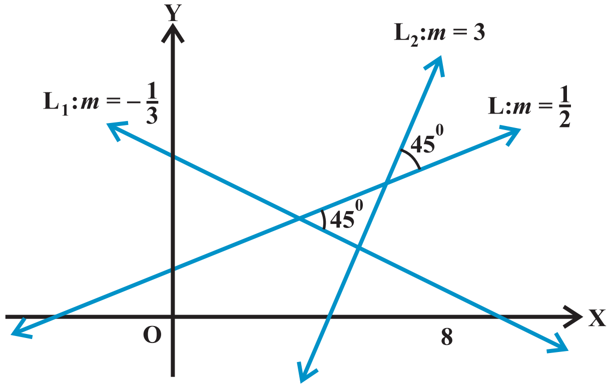

Fig 10. 7

Example 3 Line through the points (–2, 6) and (4, 8) is perpendicular to the line through the points (8, 12) and (x, 24). Find the value of x.

Solution Slope of the line through the points (– 2, 6) and (4, 8) is

$$m_1 = \frac{8-6}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$$

Slope of the line through the points (8, 12) and (x, 24) is

$$m_2 = \frac{24-12}{x-8} = \frac{12}{x-8}$$

Since two lines are perpendicular,
$m_1\, m_2 = -1$, which gives

$$\frac{1}{3} \times \frac{12}{x-8} = -1 \quad \text{or} \quad x = 4.$$

10.2.4 *Collinearity of three points*

We know that slopes of two parallel lines are equal. If two lines having the same slope pass through a common point, then two lines will coincide. Hence, if A, B and C are three points in the XY-plane, then they will lie on a line, i.e., three points are collinear (Fig 10.8) if and only if slope of AB = slope of BC.

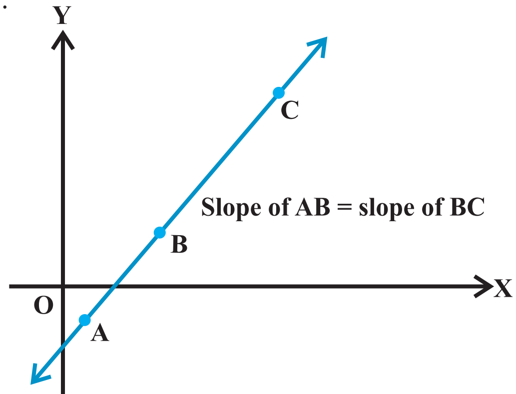

Fig 10. 8

Example 4 Three points P (h, k), Q (x_1, y_1) and R (x_2, y_2) lie on a line. Show that

$$(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$

Solution Since points P, Q and R are collinear, we have

$$\text{Slope of PQ} = \text{Slope of QR, i.e.,} \quad \frac{y_1 - k}{x_1 - h} = \frac{y_2 - y_1}{x_2 - x_1}$$

or $$\frac{k - y_1}{h - x_1} = \frac{y_2 - y_1}{x_2 - x_1},$$

or $$(h - x_1)(y_2 - y_1) = (k - y_1)(x_2 - x_1).$$

Example 5 In Fig 10.9, time and distance graph of a linear motion is given. Two positions of time and distance are recorded as, when T = 0, D = 2 and when T = 3, D = 8. Using the concept of slope, find law of motion, i.e., how distance depends upon time.

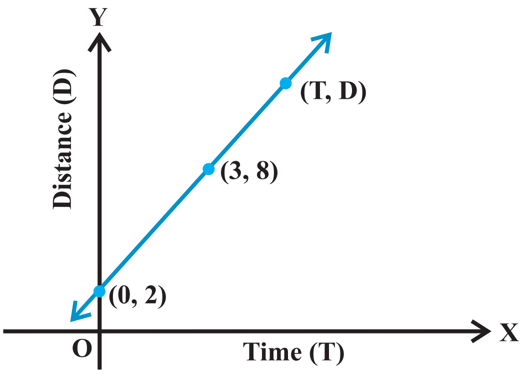

Fig 10.9

Solution Let (T, D) be any point on the line, where D denotes the distance at time T. Therefore, points (0, 2), (3, 8) and (T, D) are collinear so that

$$\frac{8-2}{3-0} = \frac{D-8}{T-3} \quad \text{or} \quad 6(T-3) = 3(D-8)$$

or $$D = 2(T + 1),$$

which is the required relation.

EXERCISE 10.1

1. Draw a quadrilateral in the Cartesian plane, whose vertices are (– 4, 5), (0, 7), (5, – 5) and (– 4, –2). Also, find its area.
2. The base of an equilateral triangle with side $2a$ lies along the y-axis such that the mid-point of the base is at the origin. Find vertices of the triangle.
3. Find the distance between P (x_1, y_1) and Q (x_2, y_2) when : (i) PQ is parallel to the y-axis, (ii) PQ is parallel to the x-axis.
4. Find a point on the x-axis, which is equidistant from the points (7, 6) and (3, 4).
5. Find the slope of a line, which passes through the origin, and the mid-point of the line segment joining the points P (0, – 4) and B (8, 0).

6. Without using the Pythagoras theorem, show that the points (4, 4), (3, 5) and (–1, –1) are the vertices of a right angled triangle.
7. Find the slope of the line, which makes an angle of 30° with the positive direction of y-axis measured anticlockwise.
8. Find the value of x for which the points (x, – 1), (2,1) and (4, 5) are collinear.
9. Without using distance formula, show that points (– 2, – 1), (4, 0), (3, 3) and (–3, 2) are the vertices of a parallelogram.
10. Find the angle between the x-axis and the line joining the points (3,–1) and (4,–2).
11. The slope of a line is double of the slope of another line. If tangent of the angle between them is $\frac{1}{3}$, find the slopes of the lines.
12. A line passes through (x_1, y_1) and (h, k). If slope of the line is m, show that
$$k - y_1 = m\,(h - x_1).$$
13. If three points $(h, 0)$, (a, b) and $(0, k)$ lie on a line, show that $\frac{a}{h} + \frac{b}{k} = 1$.
14. Consider the following population and year graph (Fig 10.10), find the slope of the line AB and using it, find what will be the population in the year 2010?

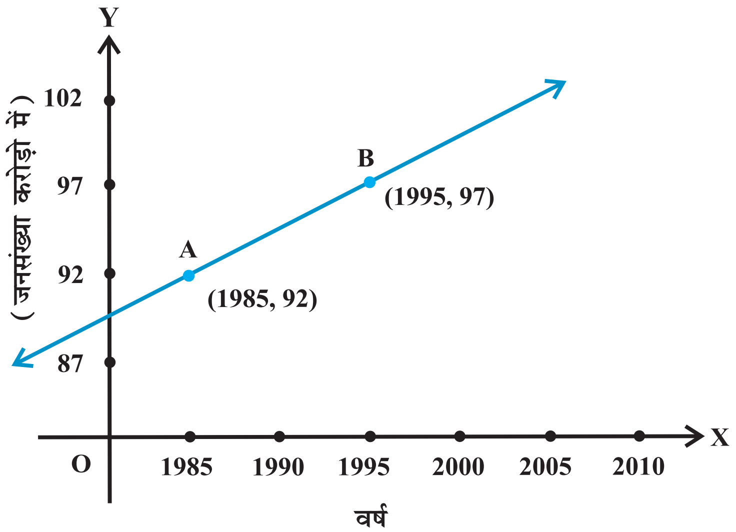

Fig 10.10

10.3 Various Forms of the Equation of a Line

We know that every line in a plane contains infinitely many points on it. This relationship between line and points leads us to find the solution of the following problem:

How can we say that a given point lies on the given line? Its answer may be that for a given line we should have a definite condition on the points lying on the line. Suppose P (x, y) is an arbitrary point in the XY-plane and L is the given line. For the equation of L, we wish to construct a *statement* or *condition* for the point P that is true, when P is on L, otherwise false. Of course the statement is merely an algebraic equation involving the variables x and y. Now, we will discuss the equation of a line under different conditions.

10.3.1 *Horizontal and vertical lines* If a horizontal line L is at a distance a from the x-axis then ordinate of every point lying on the line is either a or $-a$ [Fig 10.11 (a)]. Therefore, equation of the line L is either $y = a$ or $y = -a$. Choice of sign will depend upon the position of the line according as the line is above or below the y-axis. Similarly, the equation of a vertical line at a distance b from the x-axis is either $x = b$ or $x = -b$ [Fig 10.11(b)].

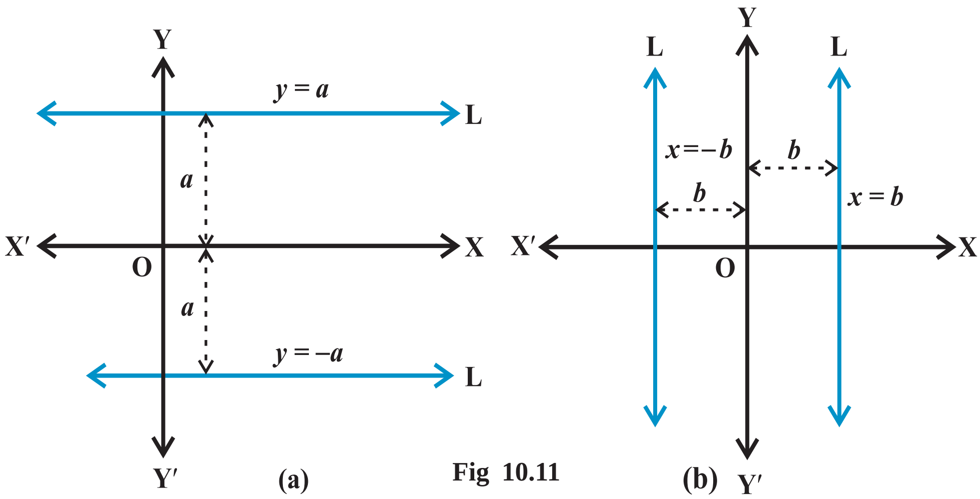

Fig 10.11

Example 6 Find the equations of the lines parallel to axes and passing through $(-2, 3)$.

Solution Position of the lines is shown in the Fig 10.12. The y-coordinate of every point on the line parallel to x-axis is 3, therefore, equation of the line parallel to x-axis and passing through $(-2, 3)$ is $y = 3$. Similarly, equation of the line parallel to y-axis and passing through $(-2, 3)$ is $x = -2$.

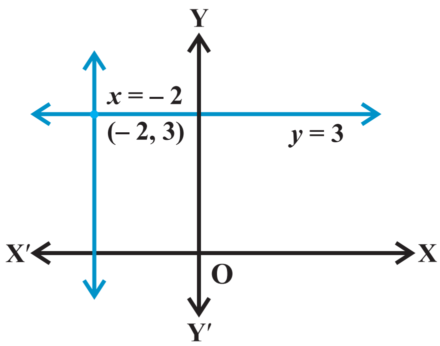

Fig 10.12

10.3.2 *Point-slope form* Suppose that $P_0(x_0, y_0)$ is a fixed point on a non-vertical line L, whose slope is m. Let P (x, y) be an arbitrary point on L (Fig 10.13).

Then, by the definition, the slope of L is given by

$$m = \frac{y - y_0}{x - x_0}, \text{ i.e., } y - y_0 = m(x - x_0) \quad ...(1)$$

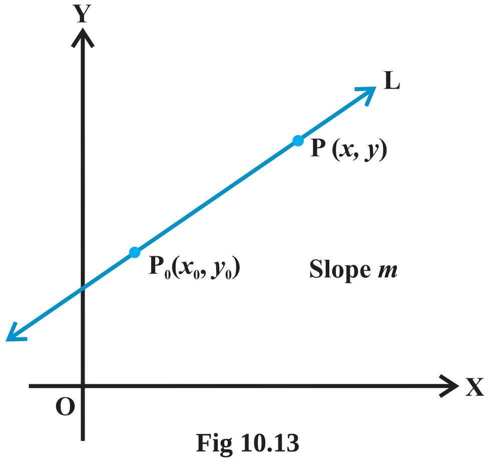

Fig 10.13

Since the point $P_0(x_0, y_0)$ along with all points (x, y) on L satisfies (1) and no other point in the plane satisfies (1). Equation (1) is indeed the equation for the given line L.

Thus, the point (x, y) lies on the line with slope m through the fixed point (x_0, y_0), if and only if, its coordinates satisfy the equation

$$y - y_0 = m(x - x_0)$$

Example 7 Find the equation of the line through (– 2, 3) with slope – 4.

Solution Here $m = -4$ and given point (x_0, y_0) is (– 2, 3).

By slope-intercept form formula (1) above, equation of the given line is

$y - 3 = -4(x + 2)$ or $4x + y + 5 = 0$, which is the required equation.

10.3.3 *Two-point form* Let the line L passes through two given points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$. Let P (x, y) be a general point on L (Fig 10.14).

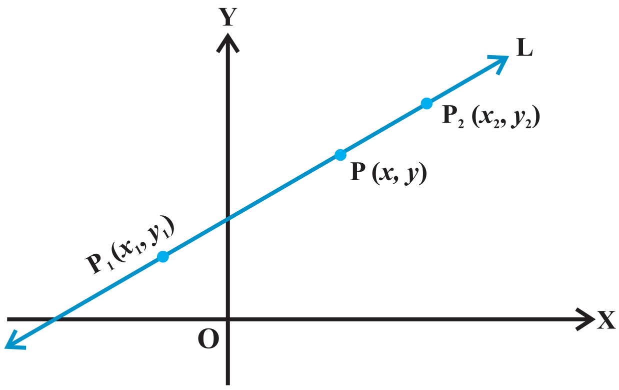

Fig 10.14

The three points P_1, P_2 and P are collinear, therefore, we have

slope of P_1P = slope of P_1P_2

i.e., $$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}, \quad \text{or} \quad y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

Thus, equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \qquad \text{... (2)}$$

Example 8 Write the equation of the line through the points (1, –1) and (3, 5).

Solution Here $x_1 = 1$, $y_1 = -1$, $x_2 = 3$ and $y_2 = 5$. Using two-point form (2) above for the equation of the line, we have

$$y - (-1) = \frac{5 - (-1)}{3 - 1}(x - 1)$$

or $\quad -3x + y + 4 = 0$, which is the required equation.

10.3.4 *Slope-intercept form* Sometimes a line is known to us with its slope and an intercept on one of the axes. We will now find equations of such lines.

Case I Suppose a line L with slope m cuts the y-axis at a distance c from the origin (Fig10.15). The distance c is called the *y-intercept* of the line L. Obviously, coordinates of the point where the line meet the y-axis are $(0, c)$. Thus, L has slope m and passes through a fixed point $(0, c)$. Therefore, by point-slope form, the equation of L is

$$y - c = m(x - 0) \quad \text{or} \quad y = mx + c$$

Thus, the point (x, y) on the line with slope m and y-intercept c lies on the line if and only if

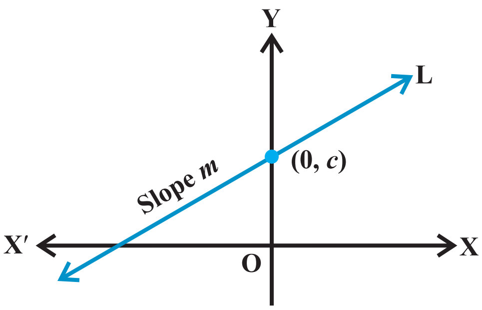

Fig 10.15

$$y = mx + c \qquad \text{...(3)}$$

Note that the value of c will be positive or negative according as the intercept is made on the positive or negative side of the y-axis, respectively.

Case II Suppose line L with slope m makes x-intercept d. Then equation of L is

$$y = m(x - d) \qquad \text{... (4)}$$

Students may derive this equation themselves by the same method as in Case I.

Example 9 Write the equation of the lines for which $\tan\theta = \frac{1}{2}$, where θ is the inclination of the line and (i) y-intercept is $-\frac{3}{2}$ (ii) x-intercept is 4.

Solution (i) Here, slope of the line is $m = \tan\theta = \frac{1}{2}$ and y - intercept $c = -\frac{3}{2}$.

Therefore, by slope-intercept form (3) above, the equation of the line is

$$y = \frac{1}{2}x - \frac{3}{2} \text{ or } 2y - x + 3 = 0,$$

which is the required equation.

(ii) Here, we have $m = \tan\theta = \frac{1}{2}$ and $d = 4$.

Therefore, by slope-intercept form (4) above, the equation of the line is

$$y = \frac{1}{2}(x - 4) \text{ or } 2y - x + 4 = 0,$$

which is the required equation.

10.3.5 *Intercept - form* Suppose a line L makes x-intercept a and y-intercept b on the axes. Obviously L meets x-axis at the point $(a, 0)$ and y-axis at the point $(0, b)$ (Fig .10.16). By two-point form of the equation of the line, we have

$$y - 0 = \frac{b-0}{0-a}(x - a) \quad \text{or} \quad ay = -bx + ab,$$

i.e.,
$$\frac{x}{a} + \frac{y}{b} = 1.$$

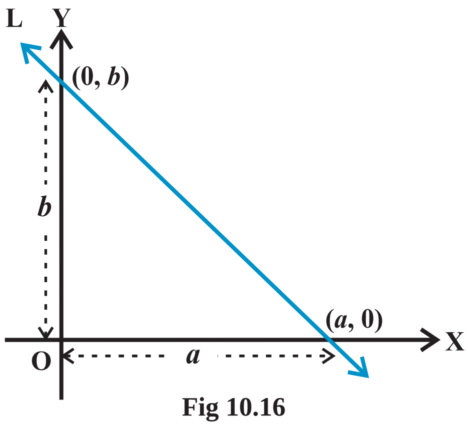

Fig 10.16

Thus, equation of the line making intercepts a and b on x-and y-axis, respectively, is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad \text{... (5)}$$

Example 10 Find the equation of the line, which makes intercepts –3 and 2 on the x- and y-axes respectively.

Solution Here $a = -3$ and $b = 2$. By intercept form (5) above, equation of the line is

$$\frac{x}{-3} + \frac{y}{2} = 1 \quad \text{or} \quad 2x - 3y + 6 = 0.$$

10.3.6 *Normal form* Suppose a non-vertical line is known to us with following data:

(i) Length of the perpendicular (normal) from origin to the line.

(ii) Angle which normal makes with the positive direction of *x*-axis.

Let L be the line, whose perpendicular distance from origin O be OA = p and the angle between the positive *x*-axis and OA be $\angle XOA = \omega$. The possible positions of line L in the Cartesian plane are shown in the Fig 10.17. Now, our purpose is to find slope of L and a point on it. Draw perpendicular AM on the *x*-axis in each case.

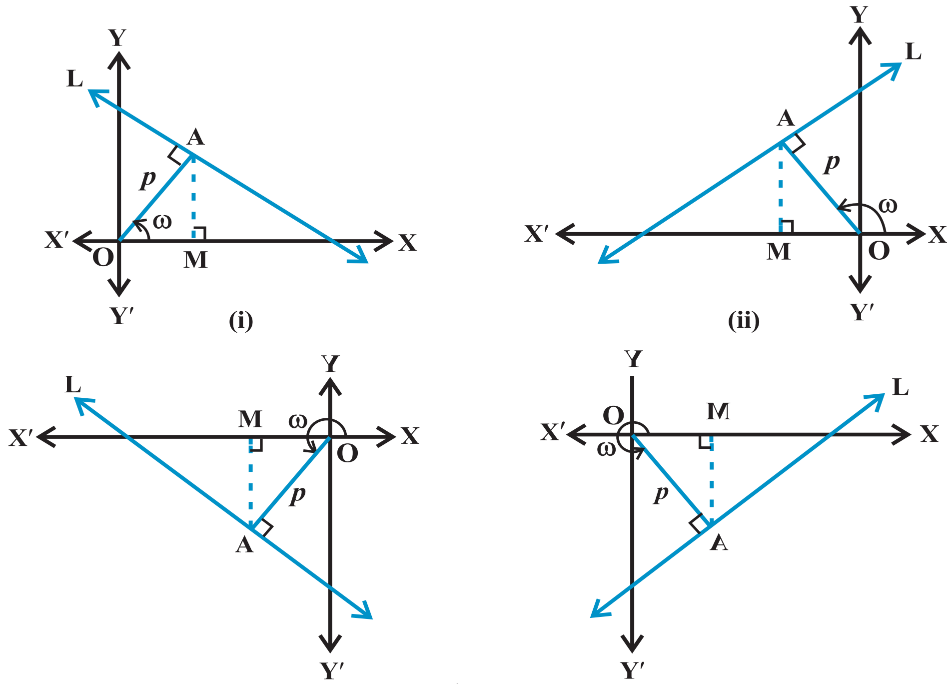

Fig 10.17

In each case, we have OM = $p \cos \omega$ and MA = $p \sin \omega$, so that the coordinates of the point A are $(p \cos \omega, p \sin \omega)$.

Further, line L is perpendicular to OA. Therefore

$$\text{The slope of the line L} = -\frac{1}{\text{slope of OA}} = -\frac{1}{\tan \omega} = -\frac{\cos \omega}{\sin \omega}.$$

Thus, the line L has slope $-\dfrac{\cos\omega}{\sin\omega}$ and point $A\left(p\cos\omega, p\sin\omega\right)$ on it. Therefore, by point-slope form, the equation of the line L is

$$y - p\sin\omega = -\frac{\cos\omega}{\sin\omega}(x - p\cos\omega) \quad \text{or} \quad x\cos\omega + y\sin\omega = p(\sin^2\omega + \cos^2\omega)$$

or $$x \cos\omega + y \sin\omega = p.$$

Hence, the equation of the line having normal distance p from the origin and angle ω which the normal makes with the positive direction of x-axis is given by

$$x \cos\omega + y \sin\omega = p \qquad \text{... (6)}$$

Example 11 Find the equation of the line whose perpendicular distance from the origin is 4 units and the angle which the normal makes with positive direction of x-axis is 15°.

Solution Here, we are given $p = 4$ and $\omega = 15^0$ (Fig10.18).

Now $$\cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

and $$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}} \quad \text{(Why?)}$$

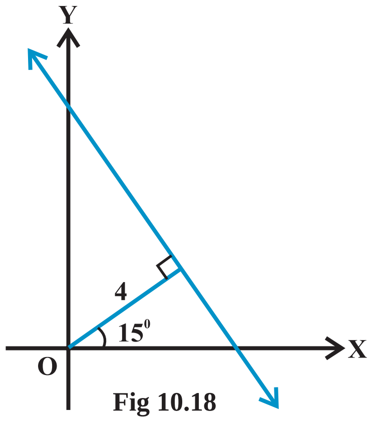

Fig 10.18

By the normal form (6) above, the equation of the line is

$$x\cos 15^0 + y\sin 15^0 = 4 \text{ or } \frac{\sqrt{3}+1}{2\sqrt{2}}x + \frac{\sqrt{3}-1}{2\sqrt{2}}y = 4 \quad \text{or} \quad \left(\sqrt{3}+1\right)x + \left(\sqrt{3}-1\right)y = 8\sqrt{2}.$$

This is the required equation.

Example 12 The Fahrenheit temperature F and absolute temperature K satisfy a linear equation. Given that K = 273 when F = 32 and that K = 373 when F = 212. Express K in terms of F and find the value of F, when K = 0.

Solution Assuming F along x-axis and K along y-axis, we have two points (32, 273) and (212, 373) in XY-plane. By two-point form, the point (F, K) satisfies the equation

$$\mathrm{K} - 273 = \frac{373-273}{212-32}(\mathrm{F}-32) \quad \text{or} \quad \mathrm{K} - 273 = \frac{100}{180}(\mathrm{F}-32)$$

or $$\mathrm{K} = \frac{5}{9}(\mathrm{F}-32) + 273 \qquad \text{... (1)}$$

which is the required relation.

When K = 0, Equation (1) gives

$$0=\frac{5}{9}(F-32)+273 \quad \text{or} \quad F-32=-\frac{273\times 9}{5}=-\ 491.4 \quad \text{or} \quad F=-459.4.$$

Alternate method We know that simplest form of the equation of a line is $y = mx + c$. Again assuming Falong *x-axis* and K along *y-axis*, we can take equation in the form

$$K = mF + c \qquad \text{... (1)}$$

Equation (1) is satisfied by (32, 273) and (212, 373). Therefore

$$273 = 32m + c \qquad \text{... (2)}$$

and

$$373 = 212m + c \qquad \text{... (3)}$$

Solving (2) and (3), we get

$$m=\frac{5}{9} \text{ and } c=\frac{2297}{9}.$$

Putting the values of m and c in (1), we get

$$K=\frac{5}{9}F+\frac{2297}{9} \qquad \text{... (4)}$$

which is the required relation. When K = 0, (4) gives F = – 459.4.

Note We know, that the equation $y = mx + c$, contains two constants, namely, m and c. For finding these two constants, we need two conditions satisfied by the equation of line. In all the examples above, we are given two conditions to determine the equation of the line.

EXERCISE 10.2

In Exercises 1 to 8, find the equation of the line which satisfy the given conditions:

1. Write the equations for the x-and y-axes.
2. Passing through the point (– 4, 3) with slope $\frac{1}{2}$.
3. Passing through (0, 0) with slope m.
4. Passing through $(2,\ 2\sqrt{3})$ and inclined with the x-axis at an angle of 75°.
5. Intersecting the x-axis at a distance of 3 units to the left of origin with slope –2.
6. Intersecting the y-axis at a distance of 2 units above the origin and making an angle of 30° with positive direction of the x-axis.
7. Passing through the points (–1, 1) and (2, – 4).

8. Perpendicular distance from the origin is 5 units and the angle made by the perpendicular with the positive x-axis is 30^0.

9. The vertices of Δ PQR are P (2, 1), Q (–2, 3) and R (4, 5). Find equation of the median through the vertex R.

10. Find the equation of the line passing through (–3, 5) and perpendicular to the line through the points (2, 5) and (–3, 6).

11. A line perpendicular to the line segment joining the points (1, 0) and (2, 3) divides it in the ratio 1: n. Find the equation of the line.

12. Find the equation of a line that cuts off equal intercepts on the coordinate axes and passes through the point (2, 3).

13. Find equation of the line passing through the point (2, 2) and cutting off intercepts on the axes whose sum is 9.

14. Find equation of the line through the point (0, 2) making an angle $\frac{2\pi}{3}$ with the positive x-axis. Also, find the equation of line parallel to it and crossing the y-axis at a distance of 2 units below the origin.

15. The perpendicular from the origin to a line meets it at the point (–2, 9), find the equation of the line.

16. The length L (in centimetrs) of a copper rod is a linear function of its Celsius temperature C. In an experiment, if L = 124.942 when C = 20 and L= 125.134 when C = 110, express L in terms of C.

17. The owner of a milk store finds that, he can sell 980 litres of milk each week at Rs 14/litre and 1220 litres of milk each week at Rs 16/litre. Assuming a linear relationship between selling price and demand, how many litres could he sell weekly at Rs 17/litre?

18. P (a, b) is the mid-point of a line segment between axes. Show that equation of the line is $\frac{x}{a}+\frac{y}{b}=2$.

19. Point R (h, k) divides a line segment between the axes in the ratio 1: 2. Find equation of the line.

20. By using the concept of equation of a line, prove that the three points (3, 0), (– 2, – 2) and (8, 2) are collinear.

10.4 General Equation of a Line

In earlier classes, we have studied general equation of first degree in two variables, Ax + By + C = 0, where A, B and C are real constants such that A and B are not zero simultaneously. Graph of the equation Ax + By + C = 0 is always a straight line.

Therefore, any equation of the form Ax + By + C = 0, where A and B are not zero simultaneously is called *general linear equation* or *general equation of a line*.

10.4.1 *Different forms of* A*x* + B*y* + *C* = 0 The general equation of a line can be reduced into various forms of the equation of a line, by the following procedures:

(a) *Slope-intercept form* If B ≠ 0, then Ax + By + C = 0 can be written as

$$y=-\frac{A}{B}x-\frac{C}{B} \text{ or } y=mx+c \qquad \ldots (1)$$

where $m=-\frac{A}{B}$ and $c=-\frac{C}{B}$.

We know that Equation (1) is the slope-intercept form of the equation of a line whose slope is $-\frac{A}{B}$, and y-intercept is $-\frac{C}{B}$.

If B = 0, then $x = -\frac{C}{A}$, which is a vertical line whose slope is undefined and x-intercept is $-\frac{C}{A}$.

(b) Intercept form If C ≠ 0, then Ax + By + C = 0 can be written as

$$\frac{x}{-\frac{C}{A}}+\frac{y}{-\frac{C}{B}}=1 \text{ or } \frac{x}{a}+\frac{y}{b}=1$$

where $a = -\frac{C}{A}$ and $b = -\frac{C}{B}$.

We know that equation (1) is intercept form of the equation of a line whose x-intercept is $-\frac{C}{A}$ and y-intercept is $-\frac{C}{B}$.

If C = 0, then Ax + By + C = 0 can be written as Ax + By = 0, which is a line passing through the origin and, therefore, has zero intercepts on the axes.

(c) Normal form Let $x \cos \omega + y \sin \omega = p$ be the normal form of the line represented by the equation Ax + By + C = 0 or Ax + By = – C. Thus, both the equations are same and therefore, $\frac{A}{\cos \omega}=\frac{B}{\sin \omega}=-\frac{C}{p}$

which gives $$\cos\omega = -\frac{Ap}{C} \text{ and } \sin\omega = -\frac{Bp}{C}.$$

Now $$\sin^2\omega + \cos^2\omega = \left(-\frac{Ap}{C}\right)^2 + \left(-\frac{Bp}{C}\right)^2 = 1$$

or $$p^2 = \frac{C^2}{A^2+B^2} \quad \text{or} \quad p = \pm\frac{C}{\sqrt{A^2+B^2}}$$

Therefore $$\cos\omega = \pm\frac{A}{\sqrt{A^2+B^2}} \text{ and } \sin\omega = \pm\frac{B}{\sqrt{A^2+B^2}}.$$

Thus, the normal form of the equation $Ax + By + C = 0$ is

$$x\cos\omega + y\sin\omega = p,$$

where $\cos\omega = \pm\frac{A}{\sqrt{A^2+B^2}}$, $\sin\omega = \pm\frac{B}{\sqrt{A^2+B^2}}$ and $p = \pm\frac{C}{\sqrt{A^2+B^2}}$.

Proper choice of signs is made so that p should be positive.

Example 13 Equation of a line is $3x - 4y + 10 = 0$. Find its (i) slope, (ii) x - and y-intercepts.

Solution (i) Given equation $3x - 4y + 10 = 0$ can be written as

$$y = \frac{3}{4}x + \frac{5}{2} \qquad \text{... (1)}$$

Comparing (1) with $y = mx + c$, we have slope of the given line as $m = \frac{3}{4}$.

(ii) Equation $3x - 4y + 10 = 0$ can be written as

$$3x - 4y = -10 \quad \text{or} \quad \frac{x}{-\frac{10}{3}} + \frac{y}{\frac{5}{2}} = 1 \qquad \text{... (2)}$$

Comparing (2) with $\frac{x}{a} + \frac{y}{b} = 1$, we have x-intercept as $a = -\frac{10}{3}$ and y-intercept as $b = \frac{5}{2}$.

Example 14 Reduce the equation $\sqrt{3}x + y - 8 = 0$ into normal form. Find the values of p and ω.

Solution Given equation is

$$\sqrt{3}x + y - 8 = 0 \qquad \text{... (1)}$$

Dividing (1) by $\sqrt{\left(\sqrt{3}\right)^2 + (1)^2} = 2$, we get

$$\frac{\sqrt{3}}{2}x + \frac{1}{2}y = 4 \quad \text{or} \quad \cos 30° \, x + \sin 30° \, y = 4 \qquad \text{... (2)}$$

Comparing (2) with $x \cos \omega + y \sin \omega = p$, we get $p = 4$ and $\omega = 30°$.

Example15 Find the angle between the lines $y - \sqrt{3}x - 5 = 0$ and $\sqrt{3}y - x + 6 = 0$.

Solution Given lines are

$$y - \sqrt{3}x - 5 = 0 \text{ or } y = \sqrt{3}x + 5 \qquad \text{... (1)}$$

and

$$\sqrt{3}y - x + 6 = 0 \text{ or } y = \frac{1}{\sqrt{3}}x - 2\sqrt{3} \qquad \text{... (2)}$$

Slope of line (1) is $m_1 = \sqrt{3}$ and slope of line (2) is $m_2 = \frac{1}{\sqrt{3}}$.

The acute angle (say) θ between two lines is given by

$$\tan\theta = \left| \frac{m_2 - m_1}{1 + m_1 m_2} \right| \qquad \text{... (3)}$$

Putting the values of m_1 and m_2 in (3), we get

$$\tan\theta = \left| \frac{\frac{1}{\sqrt{3}} - \sqrt{3}}{1 + \sqrt{3} \times \frac{1}{\sqrt{3}}} \right| = \left| \frac{1-3}{2\sqrt{3}} \right| = \frac{1}{\sqrt{3}}$$

which gives $\theta = 30°$. Hence, angle between two lines is either 30° or 180° – 30° = 150°.

Example 16 Show that two lines $a_1 x + b_1 y + c_1 = 0$ and $a_2 x + b_2 y + c_2 = 0$, where $b_1, b_2 \neq 0$ are:

(i) Parallel if $\frac{a_1}{b_1} = \frac{a_2}{b_2}$, and (ii) Perpendicular if $a_1 a_2 + b_1 b_2 = 0$.

Solution Given lines can be written as

$$y = -\frac{a_1}{b_1}x - \frac{c_1}{b_1} \quad \text{... (1)}$$

and

$$y = -\frac{a_2}{b_2}x - \frac{c_2}{b_2} \quad \text{... (2)}$$

Slopes of the lines (1) and (2) are $m_1 = -\frac{a_1}{b_1}$ and $m_2 = -\frac{a_2}{b_2}$, respectively. Now

(i) Lines are parallel, if $m_1 = m_2$, which gives

$$-\frac{a_1}{b_1} = -\frac{a_2}{b_2} \text{ or } \frac{a_1}{b_1} = \frac{a_2}{b_2}.$$

(ii) Lines are perpendicular, if $m_1 . m_2 = -1$, which gives

$$\frac{a_1}{b_1} . \frac{a_2}{b_2} = -1 \text{ or } a_1 a_2 + b_1 b_2 = 0$$

Example 17 Find the equation of a line perpendicular to the line $x - 2y + 3 = 0$ and passing through the point $(1, -2)$.

Solution Given line $x - 2y + 3 = 0$ can be written as

$$y = \frac{1}{2}x + \frac{3}{2} \quad \text{...(1)}$$

Slope of the line (1) is $m_1 = \frac{1}{2}$. Therefore, slope of the line perpendicular to line (1) is

$$m_2 = -\frac{1}{m_1} = -2$$

Equation of the line with slope -2 and passing through the point $(1, -2)$ is

$$y - (-2) = -2(x - 1) \quad \text{or} \quad y = -2x,$$

which is the required equation.

10.5 Distance of a Point From a Line

The distance of a point from a line is the length of the perpendicular drawn from the point to the line. Let L : Ax + By + C = 0 be a line, whose distance from the point P (x_1, y_1) is d. Draw a perpendicular PM from the point P to the line L (Fig10.19). If

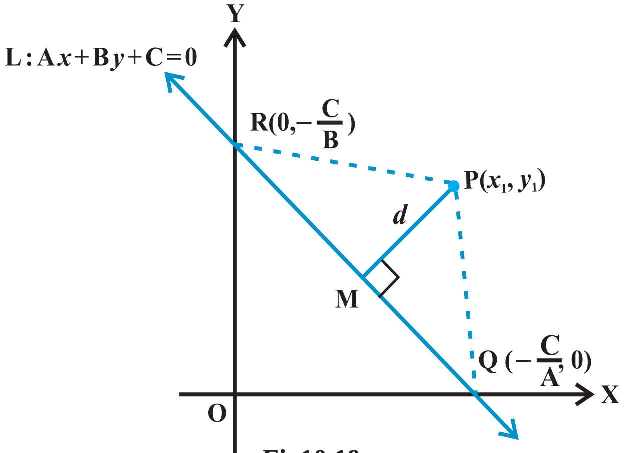

Fig10.19

the line meets the x-and y-axes at the points Q and R, respectively. Then, coordinates of the points are $Q\left(-\frac{C}{A},\ 0\right)$ and $R\left(0,\ -\frac{C}{B}\right)$. Thus, the area of the triangle PQR is given by

$$\text{area } (\Delta PQR) = \frac{1}{2} PM.QR \text{, which gives } PM = \frac{2 \text{ area } (\Delta PQR)}{QR} \qquad \ldots (1)$$

$$\text{Also, area } (\Delta PQR) = \frac{1}{2}\left| x_1\left(0+\frac{C}{B}\right) + \left(-\frac{C}{A}\right)\left(-\frac{C}{B} - y_1\right) + 0\left(y_1 - 0\right)\right|$$

$$= \frac{1}{2}\left| x_1\frac{C}{B} + y_1\frac{C}{A} + \frac{C^2}{AB}\right|$$

or $$2 \text{ area } (\Delta PQR) = \left|\frac{C}{AB}\right| . \left|Ax_1 + By_1 + C\right|, \text{ and}$$

$$QR = \sqrt{\left(0+\frac{C}{A}\right)^2 + \left(\frac{C}{B} - 0\right)^2} = \left|\frac{C}{AB}\right| \sqrt{A^2+B^2}$$

Substituting the values of area (ΔPQR) and QR in (1), we get

$$\text{PM} = \frac{|\text{A}x_1 + \text{B}y_1 + \text{C}|}{\sqrt{\text{A}^2 + \text{B}^2}}$$

or $$d = \frac{|\text{A}x_1 + \text{B}y_1 + \text{C}|}{\sqrt{\text{A}^2 + \text{B}^2}}.$$

Thus, the perpendicular distance (d) of a line Ax + By+ C = 0 from a point (x_1, y_1) is given by

$$d = \frac{|\text{A}x_1 + \text{B}y_1 + \text{C}|}{\sqrt{\text{A}^2 + \text{B}^2}}.$$

10.5.1 ***Distance between two parallel lines*** We know that slopes of two parallel lines are equal. Therefore, two parallel lines can be taken in the form

$$y = mx + c_1 \qquad \text{... (1)}$$

and $$y = mx + c_2 \qquad \text{... (2)}$$

Line (1) will intersect x-axis at the point A$\left(-\frac{c_1}{m}, 0\right)$ as shown in Fig10.20.

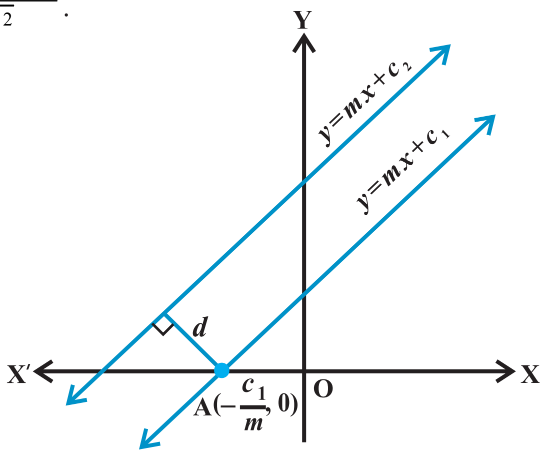

Fig10.20

Distance between two lines is equal to the length of the perpendicular from point A to line (2). Therefore, distance between the lines (1) and (2) is

$$\frac{\left|(-m)\left(-\frac{c_1}{m}\right) + (-c_2)\right|}{\sqrt{1+m^2}} \quad \text{or} \quad d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

Thus, the distance d between two parallel lines $y = mx + c_1$ and $y = mx + c_2$ is given by

$$d = \frac{|c_1 - c_2|}{\sqrt{1+m^2}}.$$

If lines are given in general form, i.e., Ax + By + C_1 = 0 and Ax + By + C_2 = 0, then above formula will take the form $d = \dfrac{|\,C_1 - C_2\,|}{\sqrt{A^2 + B^2}}$

Students can derive it themselves.

Example 18 Find the distance of the point (3, – 5) from the line $3x - 4y - 26 = 0$.

Solution Given line is $3x - 4y - 26 = 0$ ... (1)

Comparing (1) with general equation of line $Ax + By + C = 0$, we get

$$A = 3,\ B = -4 \text{ and } C = -26.$$

Given point is $(x_1, y_1) = (3, -5)$. The distance of the given point from given line is

$$d = \frac{|\,Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}} = \frac{|3.3 + (-4)(-5) - 26|}{\sqrt{3^2 + (-4)^2}} = \frac{3}{5}.$$

Example 19 Find the distance between the parallel lines $3x - 4y + 7 = 0$ and

$$3x - 4y + 5 = 0$$

Solution Here A = 3, B = –4, $C_1 = 7$ and $C_2 = 5$. Therefore, the required distance is

$$d = \frac{|\,7 - 5|}{\sqrt{3^2 + (-4)^2}} = \frac{2}{5}.$$

EXERCISE 10.3

1. Reduce the following equations into slope - intercept form and find their slopes and the y - intercepts.
   (i) $x + 7y = 0$, (ii) $6x + 3y - 5 = 0$, (iii) $y = 0$.
2. Reduce the following equations into intercept form and find their intercepts on the axes.
   (i) $3x + 2y - 12 = 0$, (ii) $4x - 3y = 6$, (iii) $3y + 2 = 0$.
3. Reduce the following equations into normal form. Find their perpendicular distances from the origin and angle between perpendicular and the positive x-axis.
   (i) $x - \sqrt{3}y + 8 = 0$, (ii) $y - 2 = 0$, (iii) $x - y = 4$.
4. Find the distance of the point (–1, 1) from the line $12(x + 6) = 5(y - 2)$.
5. Find the points on the x-axis, whose distances from the line $\dfrac{x}{3} + \dfrac{y}{4} = 1$ are 4 units.
6. Find the distance between parallel lines
   (i) $15x + 8y - 34 = 0$ and $15x + 8y + 31 = 0$ (ii) $l(x + y) + p = 0$ and $l(x + y) - r = 0$.

7. Find equation of the line parallel to the line $3x - 4y + 2 = 0$ and passing through the point (–2, 3).
8. Find equation of the line perpendicular to the line $x - 7y + 5 = 0$ and having x intercept 3.
9. Find angles between the lines $\sqrt{3}x + y = 1$ and $x + \sqrt{3}y = 1$.
10. The line through the points (h, 3) and (4, 1) intersects the line $7x - 9y - 19 = 0$. at right angle. Find the value of h.
11. Prove that the line through the point (x_1, y_1) and parallel to the line Ax + By + C = 0 is

$$A(x - x_1) + B(y - y_1) = 0.$$

12. Two lines passing through the point (2, 3) intersects each other at an angle of 60°. If slope of one line is 2, find equation of the other line.
13. Find the equation of the right bisector of the line segment joining the points (3, 4) and (–1, 2).
14. Find the coordinates of the foot of perpendicular from the point (–1, 3) to the line $3x - 4y - 16 = 0$.
15. The perpendicular from the origin to the line $y = mx + c$ meets it at the point (–1, 2). Find the values of m and c.
16. If p and q are the lengths of perpendiculars from the origin to the lines $x\cos\theta - y\sin\theta = k\cos 2\theta$ and $x \sec\theta + y \operatorname{cosec}\theta = k$, respectively, prove that $p^2 + 4q^2 = k^2$.
17. In the triangle ABC with vertices A (2, 3), B (4, –1) and C (1, 2), find the equation and length of altitude from the vertex A.
18. If p is the length of perpendicular from the origin to the line whose intercepts on the axes are a and b, then show that $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

Miscellaneous Examples

Example 20 If the lines $2x + y - 3 = 0$, $5x + ky - 3 = 0$ and $3x - y - 2 = 0$ are concurrent, find the value of k.

Solution Three lines are said to be concurrent, if they pass through a common point, i.e., point of intersection of any two lines lies on the third line. Here given lines are

$$2x + y - 3 = 0 \quad \ldots (1)$$

$$5x + ky - 3 = 0 \quad \ldots (2)$$

$$3x - y - 2 = 0 \qquad \dots (3)$$

Solving (1) and (3) by cross-multiplication method, we get

$$\frac{x}{-2-3} = \frac{y}{-9+4} = \frac{1}{-2-3} \text{ or } x = 1,\ y = 1.$$

Therefore, the point of intersection of two lines is (1, 1). Since above three lines are concurrent, the point (1, 1) will satisfy equation (3) so that

$$5.1 + k.1 - 3 = 0 \text{ or } k = -2.$$

Example 21 Find the distance of the line $4x - y = 0$ from the point P (4, 1) measured along the line making an angle of 135° with the positive x-axis.

Solution Given line is $4x - y = 0$ ... (1)

In order to find the distance of the line (1) from the point P (4, 1) along another line, we have to find the point of intersection of both the lines. For this purpose, we will first find the equation of the second line (Fig 10.21). Slope of second line is $\tan 135° = -1$. Equation of the line with slope -1 through the point P (4, 1) is

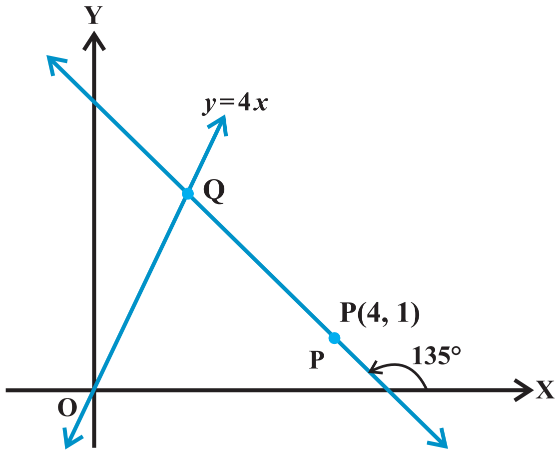

Fig 10.21

$$y - 1 = -1\,(x - 4) \text{ or } x + y - 5 = 0 \qquad \dots (2)$$

Solving (1) and (2), we get $x = 1$ and $y = 4$ so that point of intersection of the two lines is Q (1, 4). Now, distance of line (1) from the point P (4, 1) along the line (2)

= The distance between the points P (4, 1) and Q (1, 4).

$$= \sqrt{(1-4)^2 + (4-1)^2} = 3\sqrt{2} \text{ units.}$$

Example 22 Assuming that straight lines work as the plane mirror for a point, find the image of the point (1, 2) in the line $x - 3y + 4 = 0$.

Solution Let Q (h, k) is the image of the point P (1, 2) in the line

$$x - 3y + 4 = 0 \qquad \dots (1)$$

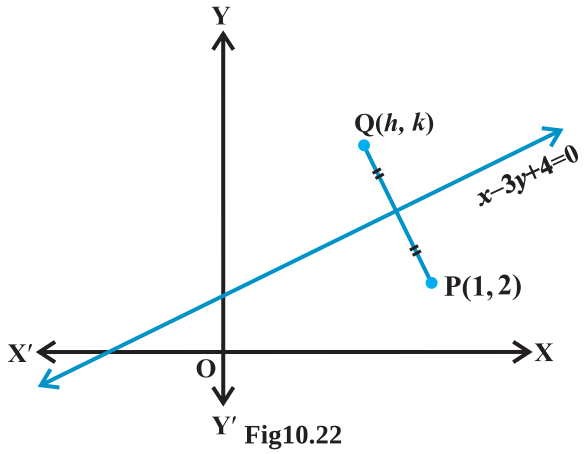

Fig10.22

Therefore, the line (1) is the perpendicular bisector of line segment PQ (Fig 10.22).

Hence Slope of line PQ $= \dfrac{-1}{\text{Slope of line } x-3y+4=0}$,

so that $\dfrac{k-2}{h-1} = \dfrac{-1}{\frac{1}{3}}$ or $3h+k=5$ ... (2)

and the mid-point of PQ, i.e., point $\left(\dfrac{h+1}{2}, \dfrac{k+2}{2}\right)$ will satisfy the equation (1) so that

$$\frac{h+1}{2} - 3\left(\frac{k+2}{2}\right) + 4 = 0 \text{ or } h - 3k = -3 \quad \text{... (3)}$$

Solving (2) and (3), we get $h = \dfrac{6}{5}$ and $k = \dfrac{7}{5}$.

Hence, the image of the point (1, 2) in the line (1) is $\left(\dfrac{6}{5}, \dfrac{7}{5}\right)$.

Example 23 Show that the area of the triangle formed by the lines

$y = m_1 x + c_1, y = m_2 x + c_2$ and $x = 0$ is $\dfrac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

Solution Given lines are

$y = m_1 x + c_1$ ... (1)

$y = m_2 x + c_2$ ... (2)

$x = 0$ ... (3)

We know that line $y = mx + c$ meets the line $x = 0$ (y-axis) at the point $(0, c)$. Therefore, two vertices of the triangle formed by lines (1) to (3) are P $(0, c_1)$ and Q $(0, c_2)$ (Fig 10. 23).

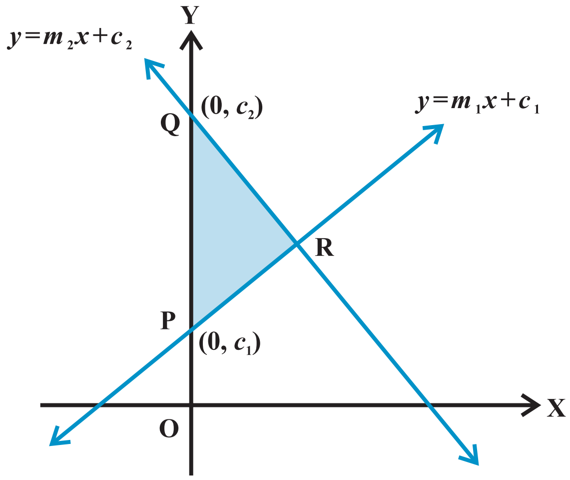

Fig 10.23

Third vertex can be obtained by solving equations (1) and (2). Solving (1) and (2), we get

$$x = \frac{(c_2 - c_1)}{(m_1 - m_2)} \text{ and } y = \frac{(m_1c_2 - m_2c_1)}{(m_1 - m_2)}$$

Therefore, third vertex of the triangle is R $\left(\frac{(c_2 - c_1)}{(m_1 - m_2)}, \frac{(m_1c_2 - m_2c_1)}{(m_1 - m_2)}\right)$.

Now, the area of the triangle is

$$= \frac{1}{2}\left| 0 \left(\frac{m_1c_2 - m_2c_1}{m_1 - m_2} - c_2\right) + \frac{c_2 - c_1}{m_1 - m_2}(c_2 - c_1) + 0\left(c_1 - \frac{m_1c_2 - m_2c_1}{m_1 - m_2}\right)\right| = \frac{(c_2 - c_1)^2}{2|m_1 - m_2|}$$

Example 24 A line is such that its segment between the lines $5x - y + 4 = 0$ and $3x + 4y - 4 = 0$ is bisected at the point (1, 5). Obtain its equation.

Solution Given lines are

$5x - y + 4 = 0$ ... (1)

$3x + 4y - 4 = 0$ ... (2)

Let the required line intersects the lines (1) and (2) at the points , (α_1, β_1) and (α_2, β_2), respectively (Fig10.24). Therefore

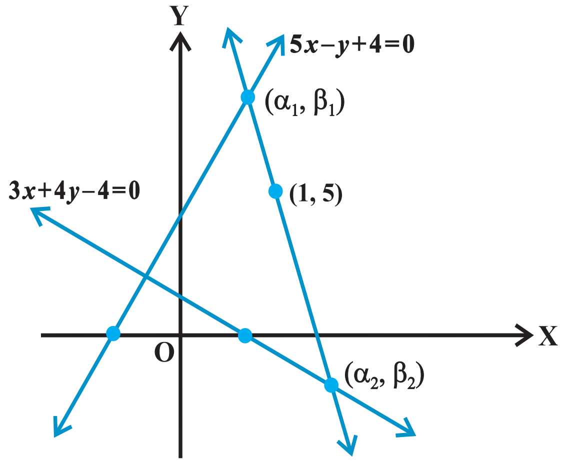

Fig 10.24

$5\alpha_1 - \beta_1 + 4 = 0$ and

$3\,\alpha_2 + 4\,\beta_2 - 4 = 0$

or $\quad \beta_1 = 5\alpha_1 + 4$ and $\beta_2 = \frac{4-3\alpha_2}{4}$.

We are given that the mid point of the segment of the required line between (α_1, β_1) and (α_2, β_2) is (1, 5). Therefore

$$\frac{\alpha_1+\alpha_2}{2} = 1 \text{ and } \frac{\beta_1+\beta_2}{2} = 5,$$

or
$$\alpha_1 + \alpha_2 = 2 \text{ and } \frac{5\alpha_1 + 4 + \frac{4-3\alpha_2}{4}}{2} = 5,$$

or $\quad \alpha_1 + \alpha_2 = 2$ and $20\alpha_1 - 3\alpha_2 = 20 \qquad \ldots (3)$

Solving equations in (3) for α_1 and α_2, we get

$$\alpha_1 = \frac{26}{23} \text{ and } \alpha_2 = \frac{20}{23} \text{ and hence, } \beta_1 = 5.\frac{26}{23} + 4 = \frac{222}{23}.$$

Equation of the required line passing through (1, 5) and (α_1, β_1) is

$$y - 5 = \frac{\beta_1 - 5}{\alpha_1 - 1}(x-1) \text{ or } y - 5 = \frac{\frac{223}{23} - 5}{\frac{26}{23} - 1}(x-1)$$

or $\quad 107x - 3y - 92 = 0$,

which is the equation of required line.

Example 25 Show that the path of a moving point such that its distances from two lines $3x - 2y = 5$ and $3x + 2y = 5$ are equal is a straight line.

Solution Given lines are

$3x - 2y = 5 \qquad \ldots (1)$

and $\quad 3x + 2y = 5 \qquad \ldots (2)$

Let (h, k) is any point, whose distances from the lines (1) and (2) are equal. Therefore

$$\frac{|3h-2k-5|}{\sqrt{9+4}} = \frac{|3h+2k-5|}{\sqrt{9+4}} \text{ or } |3h-2k-5| = |3h+2k-5|,$$

which gives $3h - 2k - 5 = 3h + 2k - 5$ or $-(3h - 2k - 5) = 3h + 2k - 5$.

Solving these two relations we get $k = 0$ or $h = \frac{5}{3}$. Thus, the point (h, k) satisfy the equations $y = 0$ or $x = \frac{5}{3}$, which represent straight lines. Hence, path of the point equidistant from the lines (1) and (2) is a straight line.

Miscellaneous Exercise on Chapter 10

1. Find the values of k for which the line $(k-3)\,x - (4 - k^2)\,y + k^2 - 7k + 6 = 0$ is
   (a) Parallel to the x-axis,
   (b) Parallel to the y-axis,
   (c) Passing through the origin.

2. Find the values of θ and p, if the equation $x \cos\theta + y \sin\theta = p$ is the normal form of the line $\sqrt{3}\,x + y + 2 = 0$.

3. Find the equations of the lines, which cut-off intercepts on the axes whose sum and product are 1 and – 6, respectively.

4. What are the points on the y-axis whose distance from the line $\frac{x}{3} + \frac{y}{4} = 1$ is 4 units.

5. Find perpendicular distance from the origin of the line joining the points $(\cos\theta, \sin\theta)$ and $(\cos\phi, \sin\phi)$.

6. Find the equation of the line parallel to y-axis and drawn through the point of intersection of the lines $x - 7y + 5 = 0$ and $3x + y = 0$.

7. Find the equation of a line drawn perpendicular to the line $\frac{x}{4} + \frac{y}{6} = 1$ through the point, where it meets the y-axis.

8. Find the area of the triangle formed by the lines $y - x = 0$, $x + y = 0$ and $x - k = 0$.

9. Find the value of p so that the three lines $3x + y - 2 = 0$, $px + 2y - 3 = 0$ and $2x - y - 3 = 0$ may intersect at one point.

10. If three lines whose equations are $y = m_1x + c_1$, $y = m_2x + c_2$ and $y = m_3x + c_3$ are concurrent, then show that $m_1(c_2 - c_3) + m_2(c_3 - c_1) + m_3(c_1 - c_2) = 0$.

11. Find the equation of the lines through the point (3, 2) which make an angle of 45° with the line $x - 2y = 3$.

12. Find the equation of the line passing through the point of intersection of the lines $4x + 7y - 3 = 0$ and $2x - 3y + 1 = 0$ that has equal intercepts on the axes.

13. Show that the equation of the line passing through the origin and making an angle θ with the line $y = mx + c$ is $\frac{y}{x} = \pm \frac{m + \tan\theta}{1 - m\tan\theta}$.

14. In what ratio, the line joining (–1, 1) and (5, 7) is divided by the line $x + y = 4$?

15. Find the distance of the line $4x + 7y + 5 = 0$ from the point (1, 2) along the line $2x - y = 0$.

16. Find the direction in which a straight line must be drawn through the point (–1, 2) so that its point of intersection with the line $x + y = 4$ may be at a distance of 3 units from this point.

17. The hypotenuse of a right angled triangle has its ends at the points (1, 3) and (– 4, 1). Find the equation of the legs (perpendicular sides) of the triangle.

18. Find the image of the point (3, 8) with respect to the line $x + 3y = 7$ assuming the line to be a plane mirror.

19. If the lines $y = 3x + 1$ and $2y = x + 3$ are equally inclined to the line $y = mx + 4$, find the value of m.

20. If sum of the perpendicular distances of a variable point P (x, y) from the lines $x + y - 5 = 0$ and $3x - 2y + 7 = 0$ is always 10. Show that P must move on a line.

21. Find equation of the line which is equidistant from parallel lines $9x + 6y - 7 = 0$ and $3x + 2y + 6 = 0$.

22. A ray of light passing through the point (1, 2) reflects on the x-axis at point A and the reflected ray passes through the point (5, 3). Find the coordinates of A.

23. Prove that the product of the lengths of the perpendiculars drawn from the points $\left(\sqrt{a^2 - b^2}, 0\right)$ and $\left(-\sqrt{a^2 - b^2}, 0\right)$ to the line $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ is b^2.

24. A person standing at the junction (crossing) of two straight paths represented by the equations $2x - 3y + 4 = 0$ and $3x + 4y - 5 = 0$ wants to reach the path whose equation is $6x - 7y + 8 = 0$ in the least time. Find equation of the path that he should follow.

Summary

- *Slope* (m) of a non-vertical line passing through the points (x_1, y_1) and (x_2, y_2) is given by $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}, \quad x_1 \neq x_2$.
- If a line makes an angle á with the positive direction of x-axis, then the slope of the line is given by $m = \tan\alpha$, $\alpha \neq 90°$.
- Slope of horizontal line is zero and slope of vertical line is undefined.

- An acute angle (say θ) between lines L_1 and L_2 with slopes m_1 and m_2 is given by $\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|, 1 + m_1 m_2 \neq 0$.
- Two lines are *parallel* if and only if their slopes are equal.
- Two lines are *perpendicular* if and only if product of their slopes is –1.
- Three points A, B and C are collinear, if and only if slope of AB = slope of BC.
- Equation of the horizontal line having distance a from the x-axis is either $y = a$ or $y = -a$.
- Equation of the vertical line having distance b from the y-axis is either $x = b$ or $x = -b$.
- The point (x, y) lies on the line with slope m and through the fixed point (x_o, y_o), if and only if its coordinates satisfy the equation $y - y_o = m\,(x - x_o)$.
- Equation of the line passing through the points (x_1, y_1) and (x_2, y_2) is given by

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

- The point (x, y) on the line with slope m and y-intercept c lies on the line if and only if $y = mx + c$.
- If a line with slope m makes x-intercept d. Then equation of the line is $y = m\,(x - d)$.
- Equation of a line making intercepts a and b on the x-and y-axis, respectively, is $\dfrac{x}{a} + \dfrac{y}{b} = 1$.
- The equation of the line having normal distance from origin p and angle between normal and the positive x-axis ω is given by $x\cos\omega + y\sin\omega = p$.
- Any equation of the form Ax + By + C = 0, with A and B are not zero, simultaneously, is called the *general linear equation* or *general equation of a line*.
- The perpendicular distance (d) of a line Ax + By+ C = 0 from a point (x_1, y_1) is given by $d = \dfrac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$.
- Distance between the parallel lines Ax + By + C_1 = 0 and Ax + By + C_2 = 0, is given by $d = \dfrac{|C_1 - C_2|}{\sqrt{A^2 + B^2}}$.