Chapter 8

BINOMIAL THEOREM

8.1 Overview:

8.1.1 An expression consisting of two terms, connected by + or – sign is called a binomial expression. For example, $x + a, 2x - 3y, \frac{1}{x} - \frac{1}{x^3}, 7x - \frac{4}{5y}$, etc., are all binomial expressions.

8.1.2 *Binomial theorem*

If a and b are real numbers and n is a positive integer, then

$$(a + b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots$$

$$\ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, b^n, \text{ where } {}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r} \text{ for } 0 \le r \le n$$

The general term or $(r + 1)^{\text{th}}$ term in the expansion is given by

$$T_{r+1} = {}^nC_r\, a^{n-r} b^r$$

8.1.3 *Some important observations*

1. The total number of terms in the binomial expansion of $(a + b)^n$ is $n + 1$, i.e. one more than the exponent n.
2. In the expansion, the first term is raised to the power of the binomial and in each subsequent terms the power of a reduces by one with simultaneous increase in the power of b by one, till power of b becomes equal to the power of binomial, i.e., the power of a is n in the first term, $(n - 1)$ in the second term and so on ending with zero in the last term. At the same time power of b is 0 in the first term, 1 in the second term and 2 in the third term and so on, ending with n in the last term.
3. In any term the sum of the indices (exponents) of 'a' and 'b' is equal to n (i.e., the power of the binomial).
4. The coefficients in the expansion follow a certain pattern known as pascal's triangle.

Index of Binomial	Coefficient of various terms
0	1
1	1 1
2	1 2 1
3	1 3 3 1
4	1 4 6 4 1
5	1 5 10 10 5 1

Each coefficient of any row is obtained by adding two coefficients in the preceding row, one on the immediate left and the other on the immediate right and each row is bounded by 1 on both sides.

The $(r+1)^{\text{th}}$ term or general term is given by

$$T_{r+1} = {}^nC_r\, a^{n-r} b^r$$

8.1.4 *Some particular cases*

If n is a positive integer, then

$$(a+b)^n = {}^nC_0\, a^n b^0 + {}^nC_1\, a^n b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + {}^nC_r\, a^{n-r} b^r + \ldots + {}^nC_n\, a^0 b^n \qquad \ldots (1)$$

In particular

1. Replacing b by $-b$ in (i), we get

   $$(a-b)^n = {}^nC_0\, a^n b^0 - {}^nC_1\, a^{n-1} b^1 + {}^nC_2\, a^{n-2} b^2 + \ldots + (-1)^r\, {}^nC_r\, a^{n-r} b^r + \ldots + (-1)^n\, {}^nC_n\, a^0 b^n \qquad \ldots (2)$$

2. Adding (1) and (2), we get

   $$(a+b)^n + (a-b)^n = 2\,[{}^nC_0\, a^n b^0 + {}^nC_2\, a^{n-2} b^2 + {}^nC_4\, a^{n-4} b^4 + \ldots]$$
   $$= 2\,[\text{terms at odd places}]$$

3. Subtracting (2) from (1), we get

   $$(a+b)^n - (a-b)^n = 2\,[{}^nC_1\, a^{n-1} b^1 + {}^nC_3\, a^{n-3} b^3 + \ldots]$$
   $$= 2\,[\text{sum of terms at even places}]$$

4. Replacing a by 1 and b by x in (1), we get

   $$(1+x)^n = {}^nC_0\, x^0 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots + {}^nC_{n-1}\, x^{n-1} + {}^nC_n\, x^n$$

   i.e. $$(1+x)^n = \sum_{r=0}^{n} {}^nC_r\, x^r$$

5. Replacing a by 1 and b by $-x$ in ... (1), we get

$$(1-x)^n = {}^nC_0\, x^0 - {}^nC_1\, x + {}^nC_2\, x^2 \ \ldots + {}^nC_{n-1}\, (-1)^{n-1}\, x^{n-1} + {}^nC_n\, (-1)^n\, x^n$$

i.e., $$(1-x)^n = \sum_{r=0}^{n} (-1)^r\, {}^nC_r\, x^r$$

8.1.5 *The p^{th} term from the end*

The p^{th} term from the end in the expansion of $(a+b)^n$ is $(n-p+2)^{\text{th}}$ term from the beginning.

8.1.6 *Middle terms*

The middle term depends upon the value of n.

(a) If n is even: then the total number of terms in the expansion of $(a+b)^n$ is $n+1$ (odd). Hence, there is only one middle term, i.e., $\left(\frac{n}{2}+1\right)^{\text{th}}$ term is the middle term.

(b) If n is odd: then the total number of terms in the expansion of $(a+b)^n$ is $n+1$ (even). So there are two middle terms i.e., $\left(\frac{n+1}{2}\right)^{\text{th}}$ and $\left(\frac{n+3}{2}\right)^{\text{th}}$ are two middle terms.

8.1.7 *Binomial coefficient*

In the Binomial expression, we have

$$(a+b)^n = {}^nC_0\, a^n + {}^nC_1\, a^{n-1}\, b + {}^nC_2\, a^{n-2}\, b^2 + \ldots + {}^nC_n\, b^n \qquad \ldots (1)$$

The coefficients ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... , ${}^nC_n$ are known as binomial or combinatorial coefficients.

Putting $a = b = 1$ in (1), we get

$${}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$$

Thus the sum of all the binomial coefficients is equal to 2^n.

Again, putting $a = 1$ and $b = -1$ in (i), we get

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots$$

Thus, the sum of all the odd binomial coefficients is equal to the sum of all the even binomial coefficients and each is equal to $\frac{2^n}{2} = 2^{n-1}$.

$${}^nC_0 + {}^nC_2 + {}^nC_4 + \ldots = {}^nC_1 + {}^nC_3 + {}^nC_5 + \ldots = 2^{n-1}$$

8.2 Solved Examples

Short Answer Type

Example 1 Find the r^{th} term in the expansion of $\left(x+\frac{1}{x}\right)^{2r}$.

Solution We have $T_r = {}^{2r}C_{r-1}\,(x)^{2r-r+1}\left(\frac{1}{x}\right)^{r-1}$

$$= \frac{\lfloor 2r}{\lfloor r-1 \lfloor r+1}\, x^{r+1-r+1}$$

$$= \frac{\lfloor 2r}{\lfloor r-1 \lfloor r+1}\, x^{2}$$

Example 2 Expand the following $(1 - x + x^2)^4$

Solution Put $1 - x = y$. Then

$$\begin{aligned}(1 - x + x^2)^4 &= (y + x^2)^4 \\ &= {}^4C_0\, y^4 (x^2)^0 + {}^4C_1\, y^3 (x^2)^1 \\ &\quad + {}^4C_2\, y^2 (x^2)^2 + {}^4C_3\, y (x^2)^3 + {}^4C_4\, (x^2)^4 \\ &= y^4 + 4y^3 x^2 + 6y^2 x^4 + 4y\, x^6 + x^8 \\ &= (1 - x)^4 + 4x^2 (1 - x)^3 + 6x^4 (1 - x)^2 + 4x^6 (1 - x) + x^8 \\ &= 1 - 4x + 10x^2 - 16x^3 + 19x^4 - 16x^5 + 10x^6 - 4x^7 + x^8\end{aligned}$$

Example 3 Find the 4^{th} term from the end in the expansion of $\left(\frac{x^3}{2}-\frac{2}{x^2}\right)^9$

Solution Since r^{th} term from the end in the expansion of $(a + b)^n$ is $(n - r + 2)^{th}$ term from the beginning. Therefore 4^{th} term from the end is $9 - 4 + 2$, i.e., 7^{th} term from the beginning, which is given by

$$T_7 = {}^9C_6\left(\frac{x^3}{2}\right)^3\left(\frac{-2}{x^2}\right)^6 = {}^9C_3\,\frac{x^9}{8}\cdot\frac{64}{x^{12}} = \frac{9\times 8\times 7}{3\times 2\times 1}\times\frac{64}{x^3} = \frac{672}{x^3}$$

Example 4 Evaluate: $\left(x^2-\sqrt{1-x^2}\right)^4+\left(x^2+\sqrt{1-x^2}\right)^4$

Solution Putting $\sqrt{1-x^2} = y$, we get

The given expression $= (x^2 - y)^4 + (x^2 + y)^4 = 2\ [x^8 + {}^4C_2\ x^4\ y^2 + {}^4C_4\ y^4]$

$$= 2\left[x^8 + \frac{4\times 3}{2\times 1}x^4\cdot(1-x^2) + (1-x^2)^2\right]$$

$$= 2\ [x^8 + 6x^4\ (1 - x^2) + (1 - 2x^2 + x^4]$$

$$= 2x^8 - 12x^6 + 14x^4 - 4x^2 + 2$$

Example 5 Find the coefficient of x^{11} in the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{12}$

Solution Let the general term, i.e., $(r + 1)^{th}$ contain x^{11}.

We have $$T_{r+1} = {}^{12}C_r\ (x^3)^{12-r}\left(-\frac{2}{x^2}\right)^r$$

$$= {}^{12}C_r\ x^{36-3r-2r}\ (-1)^r\ 2^r$$

$$= {}^{12}C_r\ (-1)^r\ 2^r\ x^{36-5r}$$

Now for this to contain x^{11}, we observe that

$$36 - 5r = 11, \text{ i.e., } r = 5$$

Thus, the coefficient of x^{11} is

$$ {}^{12}C_5\ (-1)^5\ 2^5 = -\frac{12\times 11\times 10\times 9\times 8}{5\times 4\times 3\times 2}\times 32 = -25344$$

Example 6 Determine whether the expansion of $\left(x^2 - \frac{2}{x}\right)^{18}$ will contain a term containing x^{10}?

Solution Let T_{r+1} contain x^{10}. Then

$$T_{r+1} = {}^{18}C_r\,(x^2)^{18-r}\left(\frac{-2}{x}\right)^r$$

$$= {}^{18}C_r\ x^{36-2r}\ (-1)^r\ .\ 2^r\ x^{-r}$$

$$= (-1)^r\ 2^r\ {}^{18}C_r\ x^{36-3r}$$

Thus, $$36 - 3r = 10, \text{ i.e., } r = \frac{26}{3}$$

Since r is a fraction, the given expansion cannot have a term containing x^{10}.

Example 7 Find the term independent of x in the expansion of $\left(\frac{\sqrt{x}}{\sqrt{3}}+\frac{\sqrt{3}}{2x^2}\right)^{10}$.

Solution Let $(r+1)^{\text{th}}$ term be independent of x which is given by

$$T_{r+1} = {}^{10}C_r\left(\sqrt{\frac{x}{3}}\right)^{10-r}\left(\frac{\sqrt{3}}{2x^2}\right)^r$$

$$= {}^{10}C_r\left(\frac{x}{3}\right)^{\frac{10-r}{2}} 3^{\frac{r}{2}}\left(\frac{1}{2^r\, x^{2r}}\right)$$

$$= {}^{10}C_r\, 3^{\frac{r}{2}-\frac{10-r}{2}}\, 2^{-r}\, x^{\frac{10-r}{2}-2r}$$

Since the term is independent of x, we have

$$\frac{10-r}{2}-2r = 0 \quad\Rightarrow\quad r = 2$$

Hence 3^{rd} term is independent of x and its value is given by

$$T_3 = {}^{10}C_2\frac{3^{-3}}{4} = \frac{10\times 9}{2\times 1}\times\frac{1}{9\times 12} = \frac{5}{12}$$

Example 8 Find the middle term in the expansion of $\left(2ax-\frac{b}{x^2}\right)^{12}$.

Solution Since the power of binomial is even, it has one middle term which is the $\left(\frac{12+2}{2}\right)^{\text{th}}$ term and it is given by

$$T_7 = {}^{12}C_6(2ax)^6\left(\frac{-b}{x^2}\right)^6$$

$$= {}^{12}C_6\frac{2^6a^6x^6\cdot(-b)^6}{x^{12}}$$

$$= {}^{12}C_6\frac{2^6a^6b^6}{x^6} = \frac{59136a^6b^6}{x^6}$$

Example 9 Find the middle term (terms) in the expansion of $\left(\frac{p}{x}+\frac{x}{p}\right)^9$.

Solution Since the power of binomial is odd. Therefore, we have two middle terms which are 5th and 6th terms. These are given by

$$T_5 = {}^9C_4\left(\frac{p}{x}\right)^5\left(\frac{x}{p}\right)^4 = {}^9C_4\frac{p}{x} = \frac{126p}{x}$$

and
$$T_6 = {}^9C_5\left(\frac{p}{x}\right)^4\left(\frac{x}{p}\right)^5 = {}^9C_5\frac{x}{p} = \frac{126x}{p}$$

Example 10 Show that $2^{4n+4} - 15n - 16$, where $n \in \mathbf{N}$ is divisible by 225.

Solution We have

$$\begin{aligned}
2^{4n+4} - 15n - 16 &= 2^{4(n+1)} - 15n - 16 \\
&= 16^{n+1} - 15n - 16 \\
&= (1+15)^{n+1} - 15n - 16 \\
&= {}^{n+1}C_0\,15^0 + {}^{n+1}C_1\,15^1 + {}^{n+1}C_2\,15^2 + {}^{n+1}C_3\,15^3 \\
&\quad + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1} - 15n - 16 \\
&= 1 + (n+1)\,15 + {}^{n+1}C_2\,15^2 + {}^{n+1}C_3\,15^3 \\
&\quad + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1} - 15n - 16 \\
&= 1 + 15n + 15 + {}^{n+1}C_2\,15^2 + {}^{n+1}C_3\,15^3 \\
&\quad + \ldots + {}^{n+1}C_{n+1}\,(15)^{n+1} - 15n - 16 \\
&= 15^2\,[{}^{n+1}C_2 + {}^{n+1}C_3\,15 + \ldots \text{ so on}]
\end{aligned}$$

Thus, $2^{4n+4} - 15n - 16$ is divisible by 225.

Long Answer Type

Example 11 Find numerically the greatest term in the expansion of $(2 + 3x)^9$, where

$x = \frac{3}{2}$.

Solution We have $(2+3x)^9 = 2^9\left(1+\frac{3x}{2}\right)^9$

Now,
$$\frac{T_{r+1}}{T_r} = \frac{2^9\left[{}^9C_r\left(\frac{3x}{2}\right)^r\right]}{2^9\left[{}^9C_{r-1}\left(\frac{3x}{2}\right)^{r-1}\right]}$$

$$= \frac{{}^9C_r}{{}^9C_{r-1}}\left|\frac{3x}{2}\right| = \frac{\lfloor 9}{\lfloor r\ \lfloor 9-r}\cdot\frac{\lfloor r-1\ \lfloor 10-r}{\lfloor 9}\left|\frac{3x}{2}\right|$$

$$= \frac{10-r}{r}\left|\frac{3x}{2}\right| = \frac{10-r}{r}\left(\frac{9}{4}\right) \qquad \text{Since} \quad x = \frac{3}{2}$$

Therefore,
$$\frac{T_{r+1}}{T_r} \geq 1 \Rightarrow \frac{90-9r}{4r} \geq 1$$

$$\Rightarrow 90 - 9r \geq 4r \qquad \text{(Why?)}$$

$$\Rightarrow r \leq \frac{90}{13}$$

$$\Rightarrow r \leq 6\frac{12}{13}$$

Thus the maximum value of r is 6. Therefore, the greatest term is $T_{r+1} = T_7$.

Hence,
$$T_7 = 2^9\left[{}^9C_6\left(\frac{3x}{2}\right)^6\right], \qquad \text{where } x = \frac{3}{2}$$

$$= 2^9 \cdot {}^9C_6\left(\frac{9}{4}\right)^6 = 2^9 \cdot \frac{9\times 8\times 7}{3\times 2\times 1}\left(\frac{3^{12}}{2^{12}}\right) = \frac{7\times 3^{13}}{2}$$

Example 12 If n is a positive integer, find the coefficient of x^{-1} in the expansion of $(1 + x)^n\left(1+\frac{1}{x}\right)^n$.

Solution We have

$$(1+x)^n\left(1+\frac{1}{x}\right)^n = (1+x)^n\left(\frac{x+1}{x}\right)^n = \frac{(1+x)^{2n}}{x^n}$$

Now to find the coefficient of x^{-1} in $(1+x)^n \left(1+\frac{1}{x}\right)^n$, it is equivalent to finding coefficient of x^{-1} in $\frac{(1+x)^{2n}}{x^n}$ which in turn is equal to the coefficient of x^{n-1} in the expansion of $(1+x)^{2n}$.

Since $(1+x)^{2n} = {}^{2n}C_0\, x^0 + {}^{2n}C_1\, x^1 + {}^{2n}C_2\, x^2 + \ldots + {}^{2n}C_{n-1}\, x^{n-1} + \ldots + {}^{2n}C_{2n}\, x^{2n}$

Thus the coefficient of x^{n-1} is ${}^{2n}C_{n-1}$

$$= \frac{\underline{|2n}}{\underline{|n-1}\;\underline{|2n-n+1}} = \frac{\underline{|2n}}{\underline{|n-1}\;\underline{|n+1}}$$

Example 13 Which of the following is larger?

$99^{50} + 100^{50}$ or 101^{50}

We have $(101)^{50} = (100+1)^{50}$

$$= 100^{50} + 50\,(100)^{49} + \frac{50.49}{2.1}(100)^{48} + \frac{50.49.48}{3.2.1}(100)^{47} + \ldots \quad (1)$$

Similarly $99^{50} = (100-1)^{50}$

$$= 100^{50} - 50\,.\,100^{49} + \frac{50.49}{2.1}(100)^{48} - \frac{50.49.48}{3.2.1}(100)^{47} + \ldots \quad (2)$$

Subtracting (2) from (1), we get

$$101^{50} - 99^{50} = 2\left[50\cdot(100)^{49} + \frac{50\cdot 49\cdot 48}{3\cdot 2\cdot 1}100^{47} + \ldots\right]$$

$$\Rightarrow \quad 101^{50} - 99^{50} = 100^{50} + 2\left(\frac{50\cdot 49\cdot 48}{3\cdot 2\cdot 1}\right)100^{47} + \ldots$$

$$\Rightarrow \quad 101^{50} - 99^{50} > 100^{50}$$

Hence $101^{50} > 99^{50} + 100^{50}$

Example 14 Find the coefficient of x^{50} after simplifying and collecting the like terms in the expansion of $(1+x)^{1000} + x\,(1+x)^{999} + x^2\,(1+x)^{998} + \ldots + x^{1000}$.

Solution Since the above series is a geometric series with the common ratio $\frac{x}{1+x}$, its sum is

$$\frac{(1+x)^{1000}\left[1-\left(\frac{x}{1+x}\right)^{1001}\right]}{\left[1-\left(\frac{x}{1+x}\right)\right]}$$

$$= \frac{(1+x)^{1000} - \frac{x^{1001}}{1+x}}{\frac{1+x-x}{1+x}} = (1+x)^{1001} - x^{1001}$$

Hence, coefficient of x^{50} is given by

$$^{1001}C_{50} = \frac{\underline{|1001}}{\underline{|50}\,\underline{|951}}$$

Example 15 If a_1, a_2, a_3 and a_4 are the coefficient of any four consecutive terms in the expansion of $(1 + x)^n$, prove that

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}$$

Solution Let a_1, a_2, a_3 and a_4 be the coefficient of four consecutive terms T_{r+1}, T_{r+2}, T_{r+3}, and T_{r+4} respectively. Then

a_1 = coefficient of $T_{r+1} = {}^nC_r$

a_2 = coefficient of $T_{r+2} = {}^nC_{r+1}$

a_3 = coefficient of $T_{r+3} = {}^nC_{r+2}$

and a_4 = coefficient of $T_{r+4} = {}^nC_{r+3}$

Thus $$\frac{a_1}{a_1+a_2} = \frac{^nC_r}{^nC_r + {}^nC_{r+1}}$$

$$= \frac{^nC_r}{^{n+1}C_{r+1}} \qquad (\because\ {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1})$$

$$= \frac{\lfloor n}{\lfloor r \lfloor n-r} \times \frac{\lfloor r+1 \lfloor n-r}{\lfloor n+1} = \left(\frac{r+1}{n+1}\right)$$

Similarly,
$$\frac{a_3}{a_3 + a_4} = \frac{{}^n\mathrm{C}_{r+2}}{{}^n\mathrm{C}_{r+2} + {}^n\mathrm{C}_{r+3}}$$

$$= \frac{{}^n\mathrm{C}_{r+2}}{{}^{n+1}\mathrm{C}_{r+3}} = \frac{r+3}{n+1}$$

Hence,
$$\text{L.H.S.} = \frac{a_1}{a_1 + a_2} + \frac{a_3}{a_3 + a_4} = \frac{r+1}{n+1} + \frac{r+3}{n+1} = \frac{2r+4}{n+1}$$

and
$$\text{R.H.S.} = \frac{2a_2}{a_2 + a_3} = \frac{2\left({}^n\mathrm{C}_{r+1}\right)}{{}^n\mathrm{C}_{r+1} + {}^n\mathrm{C}_{r+2}} = \frac{2\left({}^n\mathrm{C}_{r+1}\right)}{{}^{n+1}\mathrm{C}_{r+2}}$$

$$= 2\,\frac{\lfloor n}{\lfloor r+1 \lfloor n-r-1} \times \frac{\lfloor r+2 \lfloor n-r-1}{\lfloor n+1} = \frac{2\,(r+2)}{n+1}$$

Objective Type Questions (M.C.Q)

Example 16 The total number of terms in the expansion of $(x + a)^{51} - (x - a)^{51}$ after simplification is

(a) 102 (b) 25 (c) 26 (d) None of these

Solution C is the correct choice since the total number of terms are 52 of which 26 terms get cancelled.

Example 17 If the coefficients of x^7 and x^8 in $\left(2 + \frac{x}{3}\right)^n$ are equal, then n is

(a) 56 (b) 55 (c) 45 (d) 15

Solution B is the correct choice. Since $\mathrm{T}_{r+1} = {}^n\mathrm{C}_r\, a^{n-r} x^r$ in expansion of $(a + x)^n$,

Therefore,
$$\mathrm{T}_8 = {}^n\mathrm{C}_7 (2)^{n-7} \left(\frac{x}{3}\right)^7 = {}^n\mathrm{C}_7 \frac{2^{n-7}}{3^7} x^7$$

and $\quad T_9 = {}^nC_8 (2)^{n-8} \left(\frac{x}{3}\right)^8 = {}^nC_8 \frac{2^{n-8}}{3^8} x^8$

Therefore, $\quad {}^nC_7 \frac{2^{n-7}}{3^7} = {}^nC_8 \frac{2^{n-8}}{3^8}$ (since it is given that coefficient of x^7 = coefficient x^8)

$$\Rightarrow \quad \frac{\lfloor n}{\lfloor 7 \lfloor n-7} \times \frac{\lfloor 8 \lfloor n-8}{\lfloor n} = \frac{2^{n-8}}{3^8} \cdot \frac{3^7}{2^{n-7}}$$

$$\Rightarrow \quad \frac{8}{n-7} = \frac{1}{6} \quad \Rightarrow \quad n = 55$$

Example 18 If $(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$, then $a_0 + a_2 + a_4 + ... + a_{2n}$ equals.

(A) $\frac{3^n + 1}{2}$ $\qquad$ (B) $\frac{3^n - 1}{2}$ $\qquad$ (C) $\frac{1 - 3^n}{2}$ $\qquad$ (D) $3^n + \frac{1}{2}$

Solution A is the correct choice. Putting $x = 1$ and -1 in

$$(1 - x + x^2)^n = a_0 + a_1 x + a_2 x^2 + ... + a_{2n} x^{2n}$$

we get $\quad 1 = a_0 + a_1 + a_2 + a_3 + ... + a_{2n}$ ... (1)

and $\quad 3^n = a_0 - a_1 + a_2 - a_3 + ... + a_{2n}$ ... (2)

Adding (1) and (2), we get

$$3^n + 1 = 2\,(a_0 + a_2 + a_4 + ... + a_{2n})$$

Therefore $\quad a_0 + a_2 + a_4 + ... + a_{2n} = \frac{3^n + 1}{2}$

Example 19 The coefficient of x^p and x^q (p and q are positive integers) in the expansion of $(1 + x)^{p+q}$ are

(A) equal $\qquad$ (B) equal with opposite signs

(C) reciprocal of each other $\qquad$ (D) none of these

Solution A is the correct choice. Coefficient of x^p and x^q in the expansion of $(1 + x)^{p+q}$ are ${}^{p+q}C_p$ and ${}^{p+q}C_q$

and $\quad {}^{p+q}C_p = {}^{p+q}C_q = \frac{\lfloor p+q}{\lfloor p \lfloor q}$

Hence (a) is the correct answer.

Example 20 The number of terms in the expansion of $(a + b + c)^n$, where $n \in \mathbf{N}$ is

(A) $\dfrac{(n+1)(n+2)}{2}$ (B) $n + 1$

(C) $n + 2$ (D) $(n + 1)\, n$

Solution A is the correct choice. We have

$$
\begin{aligned}
(a + b + c)^n &= [a + (b + c)]^n \\
&= a^n + {}^nC_1\, a^{n-1}(b + c)^1 + {}^nC_2\, a^{n-2}(b + c)^2 \\
&\quad + \ldots + {}^nC_n (b + c)^n
\end{aligned}
$$

Further, expanding each term of R.H.S., we note that

First term consist of 1 term.

Second term on simplification gives 2 terms.

Third term on expansion gives 3 terms.

Similarly, fourth term on expansion gives 4 terms and so on.

$$
\begin{aligned}
\text{The total number of terms} &= 1 + 2 + 3 + \ldots + (n + 1) \\
&= \frac{(n+1)(n+2)}{2}
\end{aligned}
$$

Example 21 The ratio of the coefficient of x^{15} to the term independent of x in $\left(x^2 + \dfrac{2}{x}\right)^{15}$ is

(A) 12:32 (B) 1:32 (C) 32:12 (D) 32:1

Solution (B) is the correct choice. Let T_{r+1} be the general term of $\left(x^2 + \dfrac{2}{x}\right)^{15}$, so,

$$
\begin{aligned}
T_{r+1} &= {}^{15}C_r\, (x^2)^{15-r} \left(\frac{2}{x}\right)^r \\
&= {}^{15}C_r\, (2)^r\, x^{30-3r} \qquad \ldots (1)
\end{aligned}
$$

Now, for the coefficient of term containing x^{15},

$$30 - 3r = 15, \quad \text{i.e.,} \quad r = 5$$

Therefore, ${}^{15}C_5\, (2)^5$ is the coefficient of x^{15} (from (1))

To find the term independent of x, put $30 - 3r = 0$

Thus ${}^{15}C_{10}\, 2^{10}$ is the term independent of x (from (1))

Now the ratio is $\dfrac{{}^{15}C_5\, 2^5}{{}^{15}C_{10}\, 2^{10}} = \dfrac{1}{2^5} = \dfrac{1}{32}$

Example 22 If $z = \left(\dfrac{\sqrt{3}}{2} + \dfrac{i}{2}\right)^5 + \left(\dfrac{\sqrt{3}}{2} - \dfrac{i}{2}\right)^5$, then

(A) Re $(z) = 0$ (B) $I_m\,(z) = 0$

(C) Re $(z) > 0$, $I_m\,(z) > 0$ (D) Re $(z) > 0$, $I_m\,(z) < 0$

Solution B is the correct choice. On simplification, we get

$$z = 2\left[{}^5C_0\left(\frac{\sqrt{3}}{2}\right)^2 + {}^5C_2\left(\frac{\sqrt{3}}{2}\right)^3\left(\frac{i}{2}\right)^2 + {}^5C_4\left(\frac{\sqrt{3}}{2}\right)\left(\frac{i}{2}\right)^4\right]$$

Since $i^2 = -1$ and $i^4 = 1$, z will not contain any i and hence $I_m\,(z) = 0$.

8.3 EXERCISE

Short Answer Type

1. Find the term independent of x, $x \neq 0$, in the expansion of $\left(\dfrac{3x^2}{2} - \dfrac{1}{3x}\right)^{15}$.

2. If the term free from x in the expansion of $\left(\sqrt{x} - \dfrac{k}{x^2}\right)^{10}$ is 405, find the value of k.

3. Find the coefficient of x in the expansion of $(1 - 3x + 7x^2)(1 - x)^{16}$.

4. Find the term independent of x in the expansion of, $\left(3x - \dfrac{2}{x^2}\right)^{15}$.

5. Find the middle term (terms) in the expansion of

   (i) $\left(\dfrac{x}{a} - \dfrac{a}{x}\right)^{10}$ (ii) $\left(3x - \dfrac{x^3}{6}\right)^9$

6. Find the coefficient of x^{15} in the expansion of $(x - x^2)^{10}$.

7. Find the coefficient of $\frac{1}{x^{17}}$ in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$.

8. Find the sixth term of the expansion $\left(y^{\frac{1}{2}} + x^{\frac{1}{3}}\right)^n$, if the binomial coefficient of the third term from the end is 45.

[**Hint:** Binomial coefficient of third term from the end = Binomial coefficient of third term from beginning = ${}^nC_2$.]

9. Find the value of r, if the coefficients of $(2r + 4)^{\text{th}}$ and $(r - 2)^{\text{th}}$ terms in the expansion of $(1 + x)^{18}$ are equal.

10. If the coefficient of second, third and fourth terms in the expansion of $(1 + x)^{2n}$ are in A.P. Show that $2n^2 - 9n + 7 = 0$.

11. Find the coefficient of x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$.

Long Answer Type

12. If p is a real number and if the middle term in the expansion of $\left(\frac{p}{2} + 2\right)^8$ is 1120, find p.

13. Show that the middle term in the expansion of $\left(x - \frac{1}{x}\right)^{2n}$ is

$$\frac{1 \times 3 \times 5 \times \ldots (2n - 1)}{\underline{|n}} \times (-2)^n .$$

14. Find n in the binomial $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ if the ratio of 7^{th} term from the beginning to the 7^{th} term from the end is $\frac{1}{6}$.

15. In the expansion of $(x + a)^n$ if the sum of odd terms is denoted by O and the sum of

even term by E.
Then prove that
(i) $O^2 - E^2 = (x^2 - a^2)^n$ (ii) $4OE = (x + a)^{2n} - (x - a)^{2n}$

16. If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is

$$\frac{\lfloor 2n}{\left\lfloor \frac{4n-p}{3} \right. \left\lfloor \frac{2n+p}{3} \right.}.$$

17. Find the term independent of x in the expansion of $(1 + x + 2x^3)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

Objective Type Questions

Choose the correct answer from the given options in each of the Exercises 18 to 24 (M.C.Q.).

18. The total number of terms in the expansion of $(x + a)^{100} + (x - a)^{100}$ after simplification is
(A) 50 (B) 202 (C) 51 (D) none of these

19. Given the integers $r > 1$, $n > 2$, and coefficients of $(3r)^{\text{th}}$ and $(r + 2)^{\text{nd}}$ terms in the binomial expansion of $(1 + x)^{2n}$ are equal, then
(A) $n = 2r$ (B) $n = 3r$ (C) $n = 2r + 1$ (D) none of these

20. The two successive terms in the expansion of $(1 + x)^{24}$ whose coefficients are in the ratio 1:4 are
(A) 3^{rd} and 4^{th} (B) 4^{th} and 5^{th} (C) 5^{th} and 6^{th} (D) 6^{th} and 7^{th}

[**Hint**: $\frac{^{24}C_r}{^{24}C_{r+1}} = \frac{1}{4} \Rightarrow \frac{r+1}{24-r} = \frac{1}{4} \Rightarrow 4r + 4 = 24 - 4 \Rightarrow \boxed{r = 4}$]

21. The coefficient of x^n in the expansion of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ are in the ratio.
(A) 1 : 2 (B) 1 : 3 (C) 3 : 1 (D) 2 : 1

[**Hint** : $^{2n}C_n : {}^{2n-1}C_n$

22. If the coefficients of 2^{nd}, 3^{rd} and the 4^{th} terms in the expansion of $(1 + x)^n$ are in A.P., then value of n is
(A) 2 (B) 7 (c) 11 (D) 14

[**Hint:** $2\,{}^nC_2 = {}^nC_1 + {}^nC_3 \Rightarrow n^2 - 9n + 14 = 0 \Rightarrow n = 2$ or 7

23. If A and B are coefficient of x^n in the expansions of $(1 + x)^{2n}$ and $(1 + x)^{2n-1}$ respectively, then $\frac{A}{B}$ equals

(A) 1 (B) 2 (C) $\frac{1}{2}$ (D) $\frac{1}{n}$

[**Hint:** $\frac{A}{B} = \frac{{}^{2n}C_n}{{}^{2n-1}C_n} = 2$]

24. If the middle term of $\left(\frac{1}{x} + x\sin x\right)^{10}$ is equal to $7\frac{7}{8}$, then value of x is

(A) $2n\pi + \frac{\pi}{6}$ (B) $n\pi + \frac{\pi}{6}$ (C) $n\pi + (-1)^n \frac{\pi}{6}$ (D) $n\pi + (-1)^n \frac{\pi}{3}$

[**Hint:** $T_6 = {}^{10}C_5 \frac{1}{x^5} \cdot x^5 \sin^5 x = \frac{63}{8} \Rightarrow \sin^5 x = \frac{1}{2^5} \Rightarrow \sin = \frac{1}{2}$

$\Rightarrow x = n\pi + (-1)^n \frac{\pi}{6}$]

Fill in the blanks in Exercises 25 to 33.

25. The largest coefficient in the expansion of $(1 + x)^{30}$ is ________________ .

26. The number of terms in the expansion of $(x + y + z)^n$ ________________ .

[**Hint:** $(x + y + z)^n = [x + (y + z)]^n$]

27. In the expansion of $\left(x^2 - \frac{1}{x^2}\right)^{16}$, the value of constant term is ________________ .

28. If the seventh terms from the beginning and the end in the expansion of $\left(\sqrt[3]{2} + \frac{1}{\sqrt[3]{3}}\right)^n$ are equal, then n equals ________________ .

[**Hint** : $T_7 = T_{n-7+2} \Rightarrow {}^nC_6 \left(2^{\frac{1}{3}}\right)^{n-6} \left(\frac{1}{3^{\frac{1}{3}}}\right)^6 = {}^nC_{n-6} \left(2^{\frac{1}{3}}\right)^6 \left(\frac{1}{3^{\frac{1}{3}}}\right)^{n-6}$

$$\Rightarrow \left(2^{\frac{1}{3}}\right)^{n-12} = \left(\frac{1}{3^{\frac{1}{3}}}\right)^{n-12} \Rightarrow \text{only problem when } n - 12 = 0 \Rightarrow n = 12].$$

29. The coefficient of $a^{-6} b^4$ in the expansion of $\left(\frac{1}{a} - \frac{2b}{3}\right)^{10}$ is ________.

[**Hint** : $T_5 = {}^{10}C_4 \left(\frac{1}{a}\right)^b \left(\frac{-2b}{3}\right)^4 = \frac{1120}{27} a^{-6} b^4$]

30. Middle term in the expansion of $(a^3 + ba)^{28}$ is __________ .

31. The ratio of the coefficients of x^p and x^q in the expansion of $(1 + x)^{p+q}$ is __________

[**Hint:** ${}^{p+q}C_p = {}^{p+q}C_q$]

$\left(\sqrt{\frac{x}{3}} + \frac{3}{2x^2}\right)^{10}$ is

__________ .

33. If 25^{15} is divided by 13, the reminder is __________ .

State which of the statement in Exercises 34 to 40 is True or False.

34. The sum of the series $\sum_{r=0}^{10} {}^{20}C_r$ is $2^{19} + \frac{{}^{20}C_{10}}{2}$

35. The expression $7^9 + 9^7$ is divisible by 64.

Hint: $7^9 + 9^7 = (1 + 8)^7 - (1 - 8)^9$

36. The number of terms in the expansion of $[(2x + y^3)^4]^7$ is 8 **37.**

The sum of coefficients of the two middle terms in the expansion of $(1 + x)^{2n-1}$ is equal to ${}^{2n-1}C_n$.

38. The last two digits of the numbers 3^{400} are 01.

39. If the expansion of $\left(x - \frac{1}{x^2}\right)^{2n}$ contains a term independent of x, then n is a multiple of 2.

40. Number of terms in the expansion of $(a + b)^n$ where $n \in \mathbf{N}$ is one less than the power n.