# PRINCIPLE OF MATHEMATICAL INDUCTION

#### 4.1 Overview

Mathematical induction is one of the techniques which can be used to prove variety of mathematical statements which are formulated in terms of n, where n is a positive integer.

#### 4.1.1 The principle of mathematical induction

Let P(n) be a given statement involving the natural number n such that

- (i) The statement is true for n = 1, i.e., P(1) is true (or true for any fixed natural number) and
- (ii) If the statement is true for n = k (where k is a particular but arbitrary natural number), then the statement is also true for n = k + 1, i.e, truth of P(k) implies the truth of P(k + 1). Then P(n) is true for all natural numbers n.

## 4.2 Solved Examples

#### **Short Answer Type**

Prove statements in Examples 1 to 5, by using the Principle of Mathematical Induction for all  $n \in \mathbb{N}$ , that :

**Example 1** 
$$1 + 3 + 5 + ... + (2n - 1) = n^2$$

Solution Let the given statement P(n) be defined as  $P(n): 1+3+5+...+(2n-1)=n^2$ , for  $n \in \mathbb{N}$ . Note that P(1) is true, since

$$P(1): 1 = 1^2$$

Assume that P(k) is true for some  $k \in \mathbb{N}$ , i.e.,

$$P(k): 1 + 3 + 5 + ... + (2k - 1) = k^2$$

Now, to prove that P(k + 1) is true, we have

$$1 + 3 + 5 + \dots + (2k - 1) + (2k + 1)$$

$$= k^{2} + (2k + 1)$$

$$= k^{2} + 2k + 1 = (k + 1)^{2}$$
(Why?)

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all  $n \in \mathbb{N}$ .

Example 2 
$$\sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}$$
, for all natural numbers  $n \ge 2$ .

**Solution** Let the given statement P(n), be given as

$$P(n): \sum_{t=1}^{n-1} t(t+1) = \frac{n(n-1)(n+1)}{3}$$
, for all natural numbers  $n \ge 2$ .

We observe that

P(2): 
$$\sum_{t=1}^{2-1} t(t+1) = \sum_{t=1}^{1} t(t+1) = 1.2 = \frac{1.2.3}{3}$$
$$= \frac{2.(2-1)(2+1)}{3}$$

Thus, P(n) in true for n = 2.

Assume that P(n) is true for  $n = k \in \mathbb{N}$ .

e.e., 
$$P(k): \sum_{t=1}^{k-1} t(t+1) = \frac{k(k-1)(k+1)}{3}$$

To prove that P(k + 1) is true, we have

$$\sum_{t=1}^{(k+1-1)} t(t+1) = \sum_{t=1}^{k} t(t+1)$$

$$= \sum_{t=1}^{k-1} t(t+1) + k(k+1) = \frac{k(k-1)(k+1)}{3} + k(k+1)$$

$$= k(k+1) \left[ \frac{k-1+3}{3} \right] = \frac{k(k+1)(k+2)}{3}$$

$$= \frac{(k+1)((k+1)-1))((k+1)+1)}{3}$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers  $n \ge 2$ .

Example 3  $\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)...\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}$ , for all natural numbers,  $n \ge 2$ .

Solution Let the given statement be P(n), i.e.,

$$P(n): \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \cdot \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}, \text{ for all natural numbers, } n \ge 2$$

We, observe that P(2) is true, since

$$\left(1 - \frac{1}{2^2}\right) = 1 - \frac{1}{4} = \frac{4 - 1}{4} = \frac{3}{4} = \frac{2 + 1}{2 \times 2}$$

Assume that P(n) is true for some  $k \in \mathbb{N}$ , i.e.,

$$P(k): \left(1-\frac{1}{2^2}\right) \cdot \left(1-\frac{1}{3^2}\right) \cdot \cdot \cdot \left(1-\frac{1}{k^2}\right) = \frac{k+1}{2k}$$

Now, to prove that P(k + 1) is true, we have

$$\left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \cdot \left(1 - \frac{1}{k^2}\right) \cdot \left(1 - \frac{1}{(k+1)^2}\right)$$

$$= \frac{k+1}{2k} \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k^2 + 2k}{2k(k+1)} = \frac{(k+1)+1}{2(k+1)}$$

Thus, P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers,  $n \ge 2$ .

**Example 4**  $2^{2n} - 1$  is divisible by 3.

**Solution** Let the statement P(n) given as

 $P(n): 2^{2n} - 1$  is divisible by 3, for every natural number n.

We observe that P(1) is true, since

$$2^2 - 1 = 4 - 1 = 3.1$$
 is divisible by 3.

Assume that P(n) is true for some natural number k, i.e.,

P(k):  $2^{2k} - 1$  is divisible by 3, i.e.,  $2^{2k} - 1 = 3q$ , where  $q \in \mathbb{N}$ 

Now, to prove that P(k + 1) is true, we have

$$P(k+1): 2^{2(k+1)} - 1 = 2^{2k+2} - 1 = 2^{2k} \cdot 2^2 - 1$$
$$= 2^{2k} \cdot 4 - 1 = 3 \cdot 2^{2k} + (2^{2k} - 1)$$

$$= 3.2^{2k} + 3q$$
  
= 3 (2<sup>2k</sup> + q) = 3m, where  $m \in \mathbb{N}$ 

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers n.

**Example 5**  $2n + 1 < 2^n$ , for all natual numbers  $n \ge 3$ .

**Solution** Let P(n) be the given statement, i.e.,  $P(n): (2n+1) < 2^n$  for all natural numbers,  $n \ge 3$ . We observe that P(3) is true, since

$$2.3 + 1 = 7 < 8 = 2^3$$

Assume that P(n) is true for some natural number k, i.e.,  $2k + 1 < 2^k$ 

To prove P(k + 1) is true, we have to show that  $2(k + 1) + 1 < 2^{k+1}$ . Now, we have

$$2(k+1) + 1 = 2 k + 3$$
  
=  $2k + 1 + 2 < 2^k + 2 < 2^k \cdot 2 = 2^{k+1}$ .

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural numbers,  $n \ge 3$ .

## **Long Answer Type**

**Example 6** Define the sequence  $a_1$ ,  $a_2$ ,  $a_3$ ... as follows:

 $a_1 = 2$ ,  $a_n = 5$   $a_{n-1}$ , for all natural numbers  $n \ge 2$ .

- (i) Write the first four terms of the sequence.
- (ii) Use the Principle of Mathematical Induction to show that the terms of the sequence satisfy the formula  $a_n = 2.5^{n-1}$  for all natural numbers.

#### **Solution**

(i) We have  $a_1 = 2$ 

$$a_2 = 5a_{2-1} = 5a_1 = 5.2 = 10$$
  
 $a_3 = 5a_{3-1} = 5a_2 = 5.10 = 50$   
 $a_4 = 5a_{4-1} = 5a_3 = 5.50 = 250$ 

(ii) Let P(n) be the statement, i.e.,

P(n):  $a_n = 2.5^{n-1}$  for all natural numbers. We observe that P(1) is true

Assume that P(n) is true for some natural number k, i.e., P(k):  $a_k = 2.5^{k-1}$ .

Now to prove that P(k + 1) is true, we have

$$P(k + 1) : a_{k+1} = 5.a_k = 5 . (2.5^{k-1})$$
  
=  $2.5^k = 2.5^{(k+1)-1}$ 

Thus P(k + 1) is true whenever P(k) is true.

Hence, by the Principle of Mathematical Induction, P(n) is true for all natural numbers.

**Example 7** The distributive law from algebra says that for all real numbers c,  $a_1$  and  $a_2$ , we have  $c(a_1 + a_2) = ca_1 + ca_2$ .

Use this law and mathematical induction to prove that, for all natural numbers,  $n \ge 2$ , if  $c, a_1, a_2, ..., a_n$  are any real numbers, then

$$c(a_1 + a_2 + \dots + a_n) = ca_1 + ca_2 + \dots + ca_n$$

**Solution** Let P(n) be the given statement, i.e.,

 $P(n): c (a_1 + a_2 + ... + a_n) = ca_1 + ca_2 + ... ca_n$  for all natural numbers  $n \ge 2$ , for  $c, a_1, a_2, ... a_n \in \mathbf{R}$ .

We observe that P(2) is true since

$$c(a_1 + a_2) = ca_1 + ca_2$$
 (by distributive law)

Assume that P(n) is true for some natural number k, where k > 2, i.e.,

$$P(k) : c (a_1 + a_2 + ... + a_k) = ca_1 + ca_2 + ... + ca_k$$

Now to prove P(k + 1) is true, we have

$$P(k+1): c (a_1 + a_2 + ... + a_k + a_{k+1})$$

$$= c ((a_1 + a_2 + ... + a_k) + a_{k+1})$$

$$= c (a_1 + a_2 + ... + a_k) + ca_{k+1}$$
 (by distributive law)
$$= ca_1 + ca_2 + ... + ca_k + ca_{k+1}$$

Thus P(k + 1) is true, whenever P(k) is true.

Hence, by the principle of Mathematical Induction, P(n) is true for all natural numbers  $n \ge 2$ .

**Example 8** Prove by induction that for all natural number n

 $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + ... + \sin (\alpha + (n-1)\beta)$ 

$$= \frac{\sin(\alpha + \frac{n-1}{2}\beta)\sin(\frac{n\beta}{2})}{\sin(\frac{\beta}{2})}$$

**Solution** Consider P (n):  $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + ... + \sin (\alpha + (n-1)\beta)$ 

$$= \frac{\sin{(\alpha + \frac{n-1}{2}\beta)}\sin{\left(\frac{n\beta}{2}\right)}}{\sin{\left(\frac{\beta}{2}\right)}}, \text{ for all natural number } n.$$

We observe that P (1) is true, since

$$P(1) : \sin \alpha = \frac{\sin(\alpha+0)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

Assume that P(n) is true for some natural numbers k, i.e.,

$$P(k) : \sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + ... + \sin (\alpha + (k-1)\beta)$$

$$= \frac{\sin(\alpha + \frac{k-1}{2}\beta)\sin(\frac{k\beta}{2})}{\sin(\frac{\beta}{2})}$$

Now, to prove that P(k + 1) is true, we have

$$P(k+1)$$
:  $\sin \alpha + \sin (\alpha + \beta) + \sin (\alpha + 2\beta) + ... + \sin (\alpha + (k-1)\beta) + \sin (\alpha + k\beta)$ 

$$= \frac{\sin{(\alpha + \frac{k-1}{2}\beta)}\sin{\left(\frac{k\beta}{2}\right)}}{\sin{\left(\frac{\beta}{2}\right)}} + \sin{(\alpha + k\beta)}$$

$$=\frac{\sin\left(\alpha+\frac{k-1}{2}\beta\right)\sin\frac{k\beta}{2}+\sin\left(\alpha+k\beta\right)\sin\frac{\beta}{2}}{\sin\frac{\beta}{2}}$$

$$=\frac{\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+k\beta-\frac{\beta}{2}\right)+\cos\left(\alpha+k\beta-\frac{\beta}{2}\right)-\cos\left(\alpha+k\beta+\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$=\frac{\cos\left(\alpha-\frac{\beta}{2}\right)-\cos\left(\alpha+k\beta+\frac{\beta}{2}\right)}{2\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin\left(\frac{k\beta + \beta}{2}\right)}{\sin\frac{\beta}{2}}$$

$$= \frac{\sin\left(\alpha + \frac{k\beta}{2}\right)\sin(k+1)\left(\frac{\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

Thus P(k + 1) is true whenever P(k) is true.

Hence, by the Principle of Mathematical Induction P(n) is true for all natural number n.

**Example 9** Prove by the Principle of Mathematical Induction that

$$1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n+1)! - 1$$
 for all natural numbers  $n$ .

**Solution** Let P(n) be the given statement, that is,

 $P(n): 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + n \times n! = (n + 1)! - 1$  for all natural numbers n. Note that P(1) is true, since

$$P(1): 1 \times 1! = 1 = 2 - 1 = 2! - 1.$$

Assume that P(n) is true for some natural number k, i.e.,

$$P(k): 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! = (k+1)! - 1$$

To prove P(k + 1) is true, we have

$$P(k+1): 1 \times 1! + 2 \times 2! + 3 \times 3! + ... + k \times k! + (k+1) \times (k+1)!$$

$$= (k+1)! - 1 + (k+1)! \times (k+1)$$

$$= (k+1+1) (k+1)! - 1$$

$$= (k+2) (k+1)! - 1 = ((k+2)! - 1)!$$

Thus P(k+1) is true, whenever P(k) is true. Therefore, by the Principle of Mathematical Induction, P(n) is true for all natural number n.

**Example 10** Show by the Principle of Mathematical Induction that the sum  $S_n$  of the n term of the series  $1^2 + 2 \times 2^2 + 3^2 + 2 \times 4^2 + 5^2 + 2 \times 6^2$  ... is given by

$$S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{if } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{if } n \text{ is odd} \end{cases}$$

Solution Here 
$$P(n)$$
:  $S_n = \begin{cases} \frac{n(n+1)^2}{2}, & \text{when } n \text{ is even} \\ \frac{n^2(n+1)}{2}, & \text{when } n \text{ is odd} \end{cases}$ 

Also, note that any term  $T_n$  of the series is given by

$$T_n = \begin{cases} n^2 & \text{if } n \text{ is odd} \\ 2n^2 & \text{if } n \text{ is even} \end{cases}$$

We observe that P(1) is true since

P(1): 
$$S_1 = 1^2 = 1 = \frac{1.2 - 1^2 \cdot (1+1)}{2}$$

Assume that P(k) is true for some natural number k, i.e.

Case 1 When k is odd, then k + 1 is even. We have

$$P(k+1): S_{k+1} = 1^{2} + 2 \times 2^{2} + \dots + k^{2} + 2 \times (k+1)^{2}$$

$$= \frac{k^{2}(k+1)}{2} + 2 \times (k+1)^{2}$$

$$= \frac{(k+1)}{2} [k^{2} + 4(k+1)] \text{ (as } k \text{ is odd, } 1^{2} + 2 \times 2^{2} + \dots + k^{2} = k^{2} \frac{(k+1)}{2})$$

$$= \frac{k+1}{2} [k^{2} + 4k + 4]$$

$$= \frac{k+1}{2} (k+2)^{2} = (k+1) \frac{[(k+1)+1]^{2}}{2}$$

So P(k + 1) is true, whenever P(k) is true in the case when k is odd.

Case 2 When k is even, then k+1 is odd.

Now, 
$$P(k+1): 1^2 + 2 \times 2^2 + ... + 2 \cdot k^2 + (k+1)^2$$

$$= \frac{k(k+1)^2}{2} + (k+1)^2 \text{ (as } k \text{ is even, } 1^2 + 2 \times 2^2 + ... + 2k^2 = k \cdot \frac{(k+1)^2}{2})$$

$$= \frac{(k+1)^2 (k+2)}{2} = \frac{(k+1)^2 ((k+1)+1)}{2}$$

Therefore, P (k + 1) is true, whenever P (k) is true for the case when k is even. Thus P(k+1) is true whenever P(k) is true for any natural numbers k. Hence, P(n) true for all natural numbers.

## **Objective Type Questions**

Choose the correct answer in Examples 11 and 12 (M.C.Q.)

**Example 11** Let P(n): " $2^n < (1 \times 2 \times 3 \times ... \times n)$ ". Then the smallest positive integer for which P(n) is true is

**Solution** Answer is D, since

P(1): 2 < 1 is false

 $P(2): 2^2 < 1 \times 2$  is false

 $P(3): 2^3 < 1 \times 2 \times 3$  is false

But

$$P(4): 2^4 < 1 \times 2 \times 3 \times 4$$
 is true

**Example 12** A student was asked to prove a statement P(n) by induction. He proved that P (k + 1) is true whenever P (k) is true for all  $k > 5 \in \mathbb{N}$  and also that P (5) is true. On the basis of this he could conclude that P(n) is true

(A) for all 
$$n \in \mathbb{N}$$

(B) for all 
$$n > 5$$

(C) for all 
$$n \ge 5$$

(D) for all 
$$n < 5$$

**Solution** Answer is (C), since P(5) is true and P(k + 1) is true, whenever P(k) is true. Fill in the blanks in Example 13 and 14.

**Example 13** If P (n): "2.4<sup>2n+1</sup> + 3<sup>3n+1</sup> is divisible by  $\lambda$  for all  $n \in \mathbb{N}$ " is true, then the value of  $\lambda$  is \_\_\_\_

**Solution** Now, for n = 1,

$$2.4^{2+1} + 3^{3+1} = 2.4^3 + 3^4 = 2.64 + 81 = 128 + 81 = 209,$$
  
for  $n = 2, 2.4^5 + 3^7 = 8.256 + 2187 = 2048 + 2187 = 4235$ 

Note that the H.C.F. of 209 and 4235 is 11. So  $2.4^{2n+1} + 3^{3n+1}$  is divisible by 11. Hence,  $\lambda$  is 11

**Example 14** If P (n): " $49^n + 16^n + k$  is divisible by 64 for  $n \in \mathbb{N}$ " is true, then the least negative integral value of k is \_\_\_\_\_.

**Solution** For n = 1, P(1) : 65 + k is divisible by 64.

Thus k, should be -1 since, 65 - 1 = 64 is divisible by 64.

**Example 15** State whether the following proof (by mathematical induction) is true or false for the statement.

P(n): 
$$1^2 + 2^2 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

**Proof** By the Principle of Mathematical induction, P(n) is true for n = 1,

$$1^2 = 1 = \frac{1(1+1)(2\cdot 1+1)}{6}$$
. Again for some  $k \ge 1$ ,  $k^2 = \frac{k(k+1)(2k+1)}{6}$ . Now we

prove that

$$(k+1)^2 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

#### **Solution** False

Since in the inductive step both the inductive hypothesis and what is to be proved are wrong.

## 4.3 EXERCISE

## **Short Answer Type**

- 1. Give an example of a statement P(n) which is true for all  $n \ge 4$  but P(1), P(2) and P(3) are not true. Justify your answer.
- 2. Give an example of a statement P(*n*) which is true for all *n*. Justify your answer. Prove each of the statements in Exercises 3 16 by the Principle of Mathematical Induction:
- 3.  $4^n 1$  is divisible by 3, for each natural number n.
- **4.**  $2^{3n}-1$  is divisible by 7, for all natural numbers n.
- 5.  $n^3 7n + 3$  is divisible by 3, for all natural numbers n.
- **6.**  $3^{2n}-1$  is divisible by 8, for all natural numbers *n*.

- 7. For any natural number n,  $7^n 2^n$  is divisible by 5.
- **8.** For any natural number n,  $x^n y^n$  is divisible by x y, where x and y are any integers with  $x \neq y$ .
- 9.  $n^3 n$  is divisible by 6, for each natural number  $n \ge 2$ .
- **10.**  $n(n^2 + 5)$  is divisible by 6, for each natural number n.
- 11.  $n^2 < 2^n$  for all natural numbers  $n \ge 5$ .
- 12. 2n < (n+2)! for all natural number n.
- 13.  $\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \dots + \frac{1}{\sqrt{n}}$ , for all natural numbers  $n \ge 2$ .
- **14.**  $2 + 4 + 6 + ... + 2n = n^2 + n$  for all natural numbers *n*.
- **15.**  $1 + 2 + 2^2 + ... + 2^n = 2^{n+1} 1$  for all natural numbers n.
- **16.** 1+5+9+...+(4n-3)=n(2n-1) for all natural numbers n.

## Long Answer Type

Use the Principle of Mathematical Induction in the following Exercises.

- 17. A sequence  $a_1, a_2, a_3$  ... is defined by letting  $a_1 = 3$  and  $a_k = 7a_{k-1}$  for all natural numbers  $k \ge 1$ . Show that  $a_n = 3.7^{n-1}$  for all natural numbers.
- **18.** A sequence  $b_0$ ,  $b_1$ ,  $b_2$  ... is defined by letting  $b_0 = 5$  and  $b_k = 4 + b_{k-1}$  for all natural numbers k. Show that  $b_n = 5 + 4n$  for all natural number n using mathematical induction.
- 19. A sequence  $d_1$ ,  $d_2$ ,  $d_3$  ... is defined by letting  $d_1 = 2$  and  $d_k = \frac{d_{k-1}}{L}$  for all natural numbers,  $k \ge 2$ . Show that  $d_n = \frac{2}{n!}$  for all  $n \in \mathbb{N}$ .
- **20.** Prove that for all  $n \in \mathbb{N}$  $\cos \alpha + \cos (\alpha + \beta) + \cos (\alpha + 2\beta) + ... + \cos (\alpha + (n-1)\beta)$

$$= \frac{\cos\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$$

- $= \frac{\cos\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\left(\frac{n\beta}{2}\right)}{\sin\frac{\beta}{2}}$  **21.** Prove that,  $\cos\theta\cos 2\theta\cos 2\theta\cos 2\theta \ldots\cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n\sin\theta}$ , for all  $n \in \mathbb{N}$ .
- 22. Prove that,  $\sin \theta + \sin 2\theta + \sin 3\theta + ... + \sin n\theta = \frac{\frac{\sin n\theta}{2} \sin \frac{(n+1)}{2}\theta}{\sin \frac{\theta}{2}}$ , for all  $n \in \mathbb{N}$ .

- 23. Show that  $\frac{n^5}{5} + \frac{n^3}{3} + \frac{7n}{15}$  is a natural number for all  $n \in \mathbb{N}$ .
- 24. Prove that  $\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n} > \frac{13}{24}$ , for all natural numbers n > 1.
- **25.** Prove that number of subsets of a set containing *n* distinct elements is  $2^n$ , for all  $n \in \mathbb{N}$ .

## **Objective Type Questions**

Choose the correct answers in Exercises 26 to 30 (M.C.Q.).

- **26.** If  $10^n + 3.4^{n+2} + k$  is divisible by 9 for all  $n \in \mathbb{N}$ , then the least positive integral value of k is
  - (A) 5
- (B) 3
- (C) 7
- (D) 1

- **27.** For all  $n \in \mathbb{N}$ ,  $3.5^{2n+1} + 2^{3n+1}$  is divisible by
  - (A) 19
- (B) 17
- (C) 23
- (D) 25
- **28.** If  $x^n 1$  is divisible by x k, then the least positive integral value of k is
  - (A) 1
- (B) 2
- (C) 3
- (D) 4

Fill in the blanks in the following:

**29.** If  $P(n) : 2n < n!, n \in \mathbb{N}$ , then P(n) is true for all  $n \ge \underline{\hspace{1cm}}$ .

State whether the following statement is true or false. Justify.

**30.** Let P(n) be a statement and let  $P(k) \Rightarrow P(k+1)$ , for some natural number k, then P(n) is true for all  $n \in \mathbb{N}$ .